

AN APPROACH ON GENERATING FUNCTIONS
INVOLVING MULTIPLE HYPERGEOMETRIC
SERIES

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by

SULAKSHANA BAJAJ



Under the supervision of

Dr. (Mrs.) YASMEEN

Department of Mathematics

Government College Kota, Kota 324001(India)

UNIVERSITY OF KOTA, KOTA

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*This Thesis is dedicated to my
loving father, my dear husband
and the most loving son
“Suhaas”.*

For their endless love, sacrifice, support and Encouragement...

Dr.(Mrs.) Yasmeen
(M.Phil., Ph.D.)

Department of Mathematics
Govt. College Kota, Kota (Raj.)

CERTIFICATE

I feel great pleasure in certifying that the thesis entitled “AN APPROACH ON GENERATING FUNCTIONS INVOLVING MULTIPLE HYPERGEOMETRIC SERIES”, embodies a record of the results of investigations carried out by Ms. Sulakshana Bajaj under my supervision. I am satisfied with the findings of the present work.

*She has completed the residential requirements as per rules.
I recommend the submission of thesis.*

Date: _____, 2016

Dr.(Mrs.) Yasmeen
(Research Supervisor)

DECLARATION

I hereby declare that the

(i) The thesis entitled “*AN APPROACH ON GENERATING FUNCTIONS INVOLVING MULTIPLE HYPERGEOMETRIC SERIES*” submitted by me is an original piece of research work, carried out under the supervision of Dr. Yasmeen.

(ii) The above thesis has not been submitted to this university or any other university for any degree.

Date: , 2016

Sulakshana Bajaj
(Research Scholar)

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Chapter 1

An Eye On The Known Literature

1.1 Introduction

The review of literature is an indispensable part of every research work in any discipline. In the present work, it has been endeavoured to go through the renowned works of some eminent Mathematicians around the globe for the past status of the work done on the subject and the work available on it.

The scope, variety and depth of special functions and their applications are now awe-inspiring. Their rapid growth is not only in pure Mathematics and its applications to the traditional fields of Physics, Engineering and Statistics but also in new fields of applications like Behavioral Science, Optimization, Biology, Environmental Science and Economics, etc. they are emerging.

The solution of many problems related to the various areas of Science (Theoretical and Experimental Physics), Mathematical Analysis, Applied Mathematics and Cybernetic Technology, etc. is reduced to the calculations of definite and indefinite integrals or to the summation of series containing elementary and special functions.

A **special function** is a real or complex valued function of one or more real or complex variables which is specified so completely that its numerical values could be tabulated in principle. Besides elementary functions such as x^n , e^x , $\ln(x)$, and $\sin x$, higher functions, both transcendental (such as Bessel functions) and algebraic (such as various polynomials) functions come under the category of special functions. In fact special functions are solutions of a wide class of mathematically and physically relevant functional equations. As far as the origin of special functions is concerned the special function of mathematical physics arises in the solution of partial differential equations governing the behavior of certain physical quantities.

Special functions have numerous applications in astronomy, trigonometric functions which have been studied for over a thousand years. Even the series expansions for sine and cosine, as well as the arc tangent were known for long time ago from the fourteenth century. Since then the subject of special functions has been continuously developed with contribution of several mathematicians includ-

ing Euler, Legendre, Laplace, Gauss, Kummer, Riemann and Ramanujan. The discoveries of new special functions and applications of this kind of functions to new areas of mathematics have initiated a great interest of this field in the past several years.

In recent years, particular cases of long familiar special functions have been clearly defined and applied as orthogonal polynomials. The adjective special is used in this connection because here we are not, as in analysis, concerned with the general properties of functions, but only with the properties of functions which arise in the solution of special problems.

The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis. It flourished in the nineteenth century as part of the theory of complex variables. In the second half of the twentieth century it has received a new impetus from a connection with Lie groups and a connection with averages of elementary functions.

The history of special functions is closely tied to the problem of terrestrial and celestial mechanics that were solved in the eighteenth and nineteenth centuries, the boundary - value problems of electromagnetism and heat in the nineteenth, and the eigenvalue problems of quantum mechanics in the twentieth.

Seventeenth-century England was the birthplace of special functions. John Wallis at Oxford took two first steps towards the theory of the gamma function long before Euler reached it. Euler found most of the major properties of the Gamma functions around 1730. In 1772 Euler evaluated the Beta-function integral in terms of the gamma function. Only the duplication and multiplication theorems remained to be discovered by Legendre and Gauss, respectively, early in the next century.

Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. The name generating function was introduced by Laplace in 1812. Since then the theory of generating functions has been developed into various directions and found wide applications in various branches of science and technology. They are used as Z-transforms

in solving certain classes of difference equations which arise in a wide variety of problems in Operation Research. Generating functions can also be used with good effect for the determination of the asymptotic behaviour of the generated sequence $\{f_n\}_{n=0}^{\infty}$ as $n \rightarrow \infty$ by suitably adapting Darboux's method. Furthermore, the existence of a generating function for a sequence $\{f_n\}$ of numbers or functions may be useful for finding their infinite sum by summability methods.

A major development was the theory of hypergeometric series which began in a systematic way (although some important results had been found by Euler and Pfaff) with Gauss's memoir on the ${}_2F_1$ series in 1812, a memoir which was a landmark also on the path towards rigour in mathematics. The ${}_3F_2$ series was studied by Clausen (1928) and the ${}_1F_1$ series by Kummer (1836). The functions which Bessel considered in his memoir of 1824 are ${}_0F_1$ series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics, near the end of the century Appell (1880) introduced hypergeometric functions of two variables, and Lauricella generalized them to several variables in 1893.

This chapter attempts to give an introduction to the topic of study and brief review of the contributions made by some of the earlier mathematicians in this field. It will not be our endeavour to give a complete chronological overlook of all the developments in this field but only those relevant to the present work will be mentioned.

The present thesis is influenced by the work of Andrews [1], Bailey [2], Burchinal-Chaundy [10], Carlitz [12-14], Carlson [15], Chandel [16], Chandel et al. [17], Chhabra-Rusia [18], Erdélyi [21,22], Erdélyi et al. [23-27], Exton [28-42], Feldheim [43], Gauss [45], Gould [46,47], Gupta [49], Hái et al. [50], Horn [51-53], Humbert [54,55], Kampé de Fériet [58], Karlsson [60-65], Khan, M. S. [66], Khandedkar [67], Kümmer [69], Lauricella [70], Manocha, B. L. [73], Manocha, H. L. [74-75], Manocha-Sharma [76], Qureshi et al. [83], Rainville [84,85], Rice [86], Saran [88-91], Schläfli [92], Sharma, B.L. [93], Sharma et al. [94,95], Slater [98], Srivastava, H. M. [99-110], Srivastava-Buschman [111,112], Srivastava-Daoust [113,114], Srivastava-Exton [115], Srivastava-Karlsson [116], Srivastava-Manocha

[117], Srivastava-Panda [118], Srivastava-Singhal [119,120], Szegő [121], Zeitlin [126].

1.2 Gamma Function, Pochhammer Symbols and Associated Results

Throughout the present work, we use the following standard notations:

$\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \dots\} = \mathbb{Z}_0^- \setminus \{0\}$.

Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The Gamma Function:

The Gamma function, introduced by Euler in 1720 when he solved the problem of extending the factorial function to all real or complex numbers, is defined as

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re(z) > 0 \quad (1.2.1)$$

Using (1.2.1), simple integration by parts, we get a recurrence relation

$$\Gamma(z+1) = z \Gamma(z) \quad (1.2.2)$$

From (1.2.2), we get another useful result

$$\Gamma(n+1) = n! \quad n = 0, 1, 2, 3, \dots \quad (1.2.3)$$

Therefore Gamma function is the generalization of factorial function. The Gamma function has several equivalent definitions, most of which are due to Euler.

Euler's Infinite Product:

Euler's infinite product is defined as

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1} \right\} \quad (1.2.4)$$

Weierstrass Infinite Product:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right\} \quad (1.2.5)$$

where γ = Euler-Mascheroni constant = 0.577215664901532860606512

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n \right) \quad (1.2.6)$$

$$-\gamma = \Gamma'(1) = \int_0^{\infty} e^{-t} \ln(t) dt \quad (1.2.7)$$

Gauss's Limit Formula:

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(z+1)(z+2) \dots (z+n)}; \quad z \neq 0, -1, -2, \dots \quad (1.2.8)$$

If $z = 0, -1, -2, \dots$ then $|\Gamma(z)| = \infty$ or equivalently $\frac{1}{|\Gamma(z)|} = 0$

Reflection Formula of Gamma Function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} = -z \Gamma(z)\Gamma(-z) \quad (1.2.9)$$

where $z \neq 0, \pm 1, \pm 2, \dots$

Gauss Function:

$$\Pi(z) = \Gamma(z+1) \quad (1.2.10)$$

The Pochhammer Symbol:

The Pochhammer symbol (or the shifted factorial) $(\lambda)_{\nu}$ ($\lambda, \nu \in \mathbb{C}$) is defined, in terms of the familiar Gamma function, by

$$(\lambda)_{\nu} := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1) \dots (\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases} \quad (1.2.11)$$

it is being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Algebraic properties of Pochhammer's symbol

$$(a)_{p+q} = (a)_p (a+p)_q = (a)_q (a+q)_p \quad (1.2.12)$$

$$(b)_{-k} = \frac{(-1)^k}{(1-b)_k}, \quad (b \neq 0, \pm 1, \pm 2, \pm 3, \dots \text{ and } k = 1, 2, 3, \dots) \quad (1.2.13)$$

$$(n-k)! = \frac{(-1)^k (n)!}{(-n)_k}, \quad (0 \leq k \leq n) \quad (1.2.14)$$

$$(-k)_k = (-1)^k k! \quad (1.2.15)$$

$$(-n)_{mk} = \frac{(-1)^{mk} n!}{(n-mk)!} \quad ; m \in \mathbb{N} \quad (1.2.16)$$

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\left(\frac{-n}{2}\right)_k \left(\frac{-n+1}{2}\right)_k}{k!} (-x)^k = {}_2F_0 \left[\begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}; \\ -; \end{matrix} -x \right] \quad (1.2.17)$$

$$\sum_{k=0}^n \frac{(\lambda)_k (\mu)_{n-k}}{k! (n-k)!} = \frac{(\lambda+\mu)_n}{(n)!}; n \geq 0 \quad (1.2.18)$$

Binomial Coefficient:

$$\binom{n}{r} = \frac{(n)!}{(r)!(n-r)!} = \frac{(-1)^r (-n)_r}{(r)!}, \quad (0 \leq r \leq n) \quad (1.2.19)$$

Gauss's Multiplication Theorem for the Product of Gamma Functions:

[85, p.26 Th. 10; 117, p.23 (27)]

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{(mz-\frac{1}{2})} \Gamma(z) \Gamma\left(z + \frac{1}{m}\right) \Gamma\left(z + \frac{2}{m}\right) \dots \Gamma\left(z + \frac{m-1}{m}\right) \quad (1.2.20)$$

where $m = 1, 2, 3, \dots$ and $mz \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

Gauss's Multiplication Theorem for the Product of Pochhammer Symbols: [117, p.23(26)]

$$(\lambda)_{mn} = m^{mn} \prod_{j=1}^m \left(\frac{\lambda+j-1}{m} \right)_n \quad (1.2.21)$$

where $n = 0, 1, 2, 3, \dots$, $m = 1, 2, 3, \dots$

In particular, we have

$$(2n)! = 2^{2n} \left(\frac{1}{2} \right)_n n! \quad \text{and} \quad (2n+1)! = 2^{2n} \left(\frac{3}{2} \right)_n n! \quad (1.2.22)$$

Legendre's Duplication Formula:

$$\sqrt{(\pi)} \Gamma(2z) = 2^{(2z-1)} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \quad (1.2.23)$$

where $2z \in \mathbb{C} \setminus \mathbb{Z}_0^-$

Triplcation Formula:

$$2\pi \Gamma(3z) = 3^{(3z-\frac{1}{2})} \Gamma(z) \Gamma\left(z + \frac{1}{3}\right) \Gamma\left(z + \frac{2}{3}\right) \quad (1.2.24)$$

where $3z \in \mathbb{C} \setminus \mathbb{Z}_0^-$

Psi Function (Digamma Function):

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (1.2.25)$$

$$\lim_{n \rightarrow \infty} \frac{n^{(b-a)} \Gamma(a+n)}{\Gamma(b+n)} = 1 \quad (1.2.26)$$

Polygamma Functions:

$$\psi^{(n)}(z) = \frac{d^n}{dz^n} \psi(z) \quad (1.2.27)$$

1.3 Ordinary Generalized Hypergeometric Function of One Variable

A series which was beyond the ordinary geometric series

$$1 + z + z^2 + z^3 + \dots \quad (1.3.1)$$

is called hypergeometric series. John Wallis first used this term in (1655), in his work “Arithmetica Infinitorum”. In particular, he studied the series

$$1 + a + a(a+1) + a(a+1)(a+2) + \dots \quad (1.3.2)$$

After J. Wallis, many other mathematicians studied the same series, notably the Swiss Mathematician L. Euler and others but it was C.F. Gauss, a famous German Mathematician who in (1812) studied the infinite series which is the generalization of the elementary geometric series and popularly known as Gauss

hypergeometric series.

In the Gaussian hypergeometric series ${}_2F_1(a, b; c; z)$, there are two numerator parameters a, b and one denominator parameter c . A natural generalization of this series is accomplished by introducing any arbitrary number of numerator and denominator parameters.

1.3.1 Power Series Form and Convergence Conditions

The generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters is defined by [117, p.42(1)]

$${}_pF_q \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p; \\ \beta_1, \beta_2, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \dots (\beta_q)_n} \frac{z^n}{n!} \quad (1.3.3)$$

$$= {}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; z)$$

where $(\lambda)_n$ is the pochhammer symbol defined by equation (1.2.11). Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z , the numerator parameters $\alpha_1, \alpha_2, \dots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \dots, \beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \dots; \quad j = 1, 2, \dots, q \quad (1.3.4)$$

Series (1.3.3) is a non-terminating series and also known as the generalized Gauss series, or simply, the generalized hypergeometric series. Supposing that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the usual restriction (1.3.4), the ${}_pF_q$ series in the definition (1.3.3)

- (i) converges for $|z| < \infty$ if $p \leq q$,
- (ii) converges for $|z| < 1$, if $p = q + 1$,
- (iii) diverges for all $z, z \neq 0$, if $p > q + 1$.

Furthermore, if we set

$$\omega = \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j, \quad (1.3.5)$$

It is known that the ${}_pF_q$ series, with $p = q + 1$, is

- (I) absolutely convergent for $|z| = 1$, if $\Re(\omega) > 0$,

- (II) conditionally convergent for $|z| = 1$, $|z| \neq 1$, if $-1 < \Re(\omega) \leq 0$,
 (III) divergent for $|z| = 1$, if $\Re(\omega) \leq -1$.

The Gauss hypergeometric function [117, p.29 (1.2.4)] is defined by the power series

$${}_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (|z| < 1; \quad a, b, c \neq 0, -1, -2, \dots) \quad (1.3.6)$$

When $a = -p$ (p being a positive integer) then we get a hypergeometric polynomial in terminating form

$${}_2F_1[-p, b; c; z] = \sum_{n=0}^p \frac{(-p)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (b, c \neq 0, -1, -2, \dots) \quad (1.3.7)$$

When $a = -p$ and $b = -q$ (p, q being positive integers) then we obtain a hypergeometric polynomial in another terminating form

$${}_2F_1[-p, -q; c; z] = \sum_{n=0}^{\min(p,q)} \frac{(-p)_n (-q)_n}{(c)_n} \frac{z^n}{n!}, \quad (c \neq 0, -1, -2, \dots) \quad (1.3.8)$$

1.3.2 Results of Generalized Hypergeometric Function:

Gauss's Summation Theorem: [85, p.49 (Th. 18)]

$${}_2F_1 \left[\begin{matrix} a, b; \\ c; \end{matrix} 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (1.3.9)$$

where $\operatorname{Re}(c-a-b) > 0$ and $c \neq 0, -1, -2, \dots$

Binomial Theorem:

In terms of hypergeometric function

$$(1-t)^{-\alpha} = {}_1F_0 \left[\begin{matrix} \alpha; \\ -; \end{matrix} t \right] = \sum_{n=0}^{\infty} (\alpha)_n \frac{t^n}{n!}, \quad |t| < 1 \quad (1.3.10)$$

$$\sum_{k=0}^n \frac{(-n)_k}{k!} = {}_1F_0 \left[\begin{matrix} -n; \\ ; \end{matrix} 1 \right] = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases} \quad (1.3.11)$$

here n is a positive integer.

Kummer's series:- An important special case is obtained taking $p = q = 1$ in

equation (1.3.3), the equation (1.3.3) reduces to the confluent hypergeometric series ${}_1F_1$ named as Kummer's series given by

$${}_1F_1(\alpha; \beta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \frac{z^n}{n!}. \quad (1.3.12)$$

Saalschütz' summation theorem for terminating series [85,p.87 Th.29]

$${}_3F_2(-N, A, B; C, D; 1) = \frac{(C-A)_N (C-B)_N}{(C)_N (C-A-B)_N} \quad (1.3.13)$$

where $(C+D) - (-N+A+B) = 1$, and N is a non-negative integer.

$${}_3F_2 \left[\begin{matrix} A, B, C ; \\ A-n, B+1; \end{matrix} \middle| 1 \right] = \frac{(1+B-A)_n \Gamma(1-C) \Gamma(1+B)}{(1-A)_n \Gamma(1+B-C)} \quad (1.3.14)$$

$Re(C) < (1-n)$

1.4 Fox-Wright Generalized Hypergeometric Function of One Variable

In 1933, E. M. Wright defined a more interesting generalized hypergeometric function of one variable [116, p.21(1.2.38, 1.2.40); 117, pp. 50-51(1.5.21), p.179 (34iii), p.395 (23)] and further generalizations of the series ${}_pF_q$ were given by Fox[44] and Wright [124, p.287; 125].

$$\begin{aligned} {}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} \middle| z \right] &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \Gamma(\alpha_2 + A_2 n) \dots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \Gamma(\beta_2 + B_2 n) \dots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!} \\ &= H_{p,q+1}^{1,p} \left[-z \middle| \begin{matrix} (1-\alpha_1, A_1), & \dots, & (1-\alpha_p, A_p) \\ (0, 1)(1-\beta_1, B_1), \dots, (1-\beta_q, B_q) \end{matrix} \right] \end{aligned} \quad (1.4.1)$$

where the coefficients $A_1, A_2, \dots, A_p, B_1, B_2, \dots, B_q$ are positive numbers and $\alpha_1, \alpha_2, \dots, \alpha_p, \beta_1, \beta_2, \dots, \beta_q$ are complex parameters. The function $H_{p,q}^{m,n}$ was given by C. Fox [44] in 1961 and Braaksma [3, p.278] gave the convergence condition.

$${}_p\Psi_q^* \left[\begin{matrix} ((\alpha_p, A_p)); \\ ((\beta_q, B_q)); \end{matrix} \middle| z \right] = {}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{nA_1} \dots (\alpha_p)_{nA_p}}{(\beta_1)_{nB_1} \dots (\beta_q)_{nB_q}} \frac{z^n}{n!} \quad (1.4.2)$$

The Fox's H-function on the RHS of (1.4.1) makes sense when either

$$\delta = (1 + B_1 + B_2 + \dots + B_q) - (A_1 + A_2 + \dots + A_p) > 0$$

and

$$0 < |z| < \infty; \quad z \neq 0$$

The equality holds only for suitably constrained values of $|z|$ or appropriately bounded $|z|$ i.e. $\delta = 0$ and $0 < |z| < R = A_1^{-A_1} A_2^{-A_2} \dots A_p^{-A_p} B_1^{B_1} B_2^{B_2} \dots B_q^{B_q}$

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, 1), \dots, (\alpha_p, 1); \\ (\beta_1, 1), \dots, (\beta_q, 1); \end{matrix} z \right] = \frac{\prod_{j=1}^p \Gamma(\alpha_j)}{\prod_{j=1}^q \Gamma(\beta_j)} {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] \quad (1.4.3)$$

Relation between Wright's Function and Fox-Wright's Hypergeometric Function[117, p.395, Q.23; p.50, Q.21]

$${}_r\Psi_s^* \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_r, A_r); \\ (\beta_1, B_1), \dots, (\beta_s, B_s); \end{matrix} z \right] = \frac{\prod_{j=1}^s \Gamma(b_j)}{\prod_{j=1}^r \Gamma(a_j)} {}_r\Psi_s \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_r, A_r); \\ (\beta_1, B_1), \dots, (\beta_s, B_s); \end{matrix} z \right] \quad (1.4.4)$$

Where $A_1, \dots, A_r, B_1, \dots, B_s$ are greater than zero; $1 + B_1 + \dots + B_s - (A_1 + \dots + A_r) \geq 0$. The equality holds only for suitably constrained values of $|z|$.

1.5 Truncated Ordinary Hypergeometric Series

The truncated generalized Gaussian hypergeometric function of one variable is defined by:

$${}_mF_n \left[\begin{matrix} (a_m); \\ (b_n); \end{matrix} y \right] \text{ to } (i+1) \text{ terms} = {}_mF_n \left[\begin{matrix} (a_m); \\ (b_n); \end{matrix} y \right]_i = \sum_{k=0}^i \frac{[(a_m)]_k}{[(b_n)]_k} \frac{y^k}{k!} \quad (1.5.1)$$

where the suffix i indicates that only first $(i+1)$ terms of the F series are to be included in the expansion [13, p.437; 71, pp. 349-350; 72, p. 394(2); 80, p. 792; 81, p.430; 98, pp.83-84 (2.6.1.1, 2.6.1.7, 2.6.1.9)] and (h_r) represents the array of r parameters given by $h_1, h_2, \dots, h_r$; $[(h_r)]_p$ means the product of r Pochhammer's symbols $(h_1)_p (h_2)_p \dots (h_r)_p$.

1.6 Appell's Double Hypergeometric Functions

The Appell's double hypergeometric function of first kind F_1 [37, p.23 (1.4.1); 117, p.53 (4)] is given by

$$F_1[a, b, c; d; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.6.1)$$

$$F_1[a, b, c; d; x, y] = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(d)_m} {}_2F_1 \left[\begin{matrix} a+m, c; \\ d+m; \end{matrix} y \right] \frac{x^m}{m!} \quad \max\{|x|, |y|\} < 1 \quad (1.6.2)$$

The Appell's double hypergeometric function of second kind F_2 [117, p.53 (5)]

$$F_2[a, b, c; d, g; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(g)_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.6.3)$$

$$F_2[a, b, c; d, g; x, y] = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(d)_m} {}_2F_1 \left[\begin{matrix} a+m, c; \\ g; \end{matrix} y \right] \frac{x^m}{m!} \quad \{|x| + |y|\} < 1 \quad (1.6.4)$$

The Appell's double hypergeometric function of third kind F_3 [117, p.53 (6)]

$$F_3[a, b, c, d; g; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n(c)_m(d)_n}{(g)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.6.5)$$

$$F_3[a, b, c, d; g; x, y] = \sum_{m=0}^{\infty} \frac{(a)_m(c)_m}{(g)_m} {}_2F_1 \left[\begin{matrix} b, d; \\ g+m; \end{matrix} y \right] \frac{x^m}{m!} \quad \max\{|x|, |y|\} < 1 \quad (1.6.6)$$

The Appell's function of fourth kind F_4 [117, p.53, (7)] is defined by

$$F_4[a, b, c, d; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad (1.6.7)$$

$$F_4[a, b, c, d; x, y] = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} {}_2F_1 \left[\begin{matrix} a+m, b+m; \\ d; \end{matrix} y \right] \frac{x^m}{m!} \quad (1.6.8)$$

$$(\sqrt{|x|} + \sqrt{|y|}) < 1; \quad a, b, c, d \neq 0, -1, -2, \dots$$

When $a = -r$ (r being a positive integer) then we have Appell's polynomial of fourth kind

$$F_4[-r; b, c, d; x, y] = \sum_{m,n=0}^{m+n \leq r} \frac{(-r)_{m+n}(b)_{m+n}}{(c)_m(d)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad (b, c, d \neq 0, -1, -2, \dots) \quad (1.6.9)$$

1.7 Humbert's Double Hypergeometric Functions

Seven confluent forms of four Appell functions were defined in 1920 by P. Humbert [55]; see also [117, p.58 (36), (37), (40), (41), (42), (43), (44)]

$$\Phi_1[a, b; c; x, y] = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s} (b)_r}{(c)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!} \quad (1.7.1)$$

where $|x| < 1$, $|y| < \infty$

$$\Phi_2[b, c; d; x, y] = \sum_{r,s=0}^{\infty} \frac{(b)_r (c)_s}{(d)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!} \quad (1.7.2)$$

where $|x| < \infty$, $|y| < \infty$

$$\Phi_3[b; d; x, y] = \sum_{r,s=0}^{\infty} \frac{(b)_r}{(d)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!} \quad (1.7.3)$$

where $|x| < \infty$, $|y| < \infty$.

$$\Psi_1[a, b; c, d; x, y] = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s} (b)_r}{(c)_r (d)_s} \frac{x^r}{r!} \frac{y^s}{s!} \quad (1.7.4)$$

where $|x| < 1$, $|y| < \infty$.

$$\Psi_2[a; c, d; x, y] = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}}{(c)_r (d)_s} \frac{x^r}{r!} \frac{y^s}{s!} \quad (1.7.5)$$

where $|x| < \infty$, $|y| < \infty$.

$$\Xi_1[a, b, c; d; x, y] = \sum_{r,s=0}^{\infty} \frac{(a)_r (b)_s (c)_r}{(d)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!} \quad (1.7.6)$$

where $|x| < 1$, $|y| < \infty$.

$$\Xi_2[a, b; c; x, y] = \sum_{r,s=0}^{\infty} \frac{(a)_r (b)_r}{(c)_{r+s}} \frac{x^r}{r!} \frac{y^s}{s!} \quad (1.7.7)$$

where $|x| < 1$, $|y| < \infty$.

1.8 Kampé de Fériet's Double Hypergeometric Functions

Kampé de Fériet's double hypergeometric function of higher order in the modified notation of Srivastava and Panda [118, pp. 423(26), 424(27)], is given by

$$F_{j:m;n}^{p;q;k} \left[\begin{matrix} (a_p) : (b_q); (d_k) ; \\ (g_j) : (e_m); (h_n); \end{matrix} ; x, y \right] = \sum_{r,s=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_{r+s} \prod_{i=1}^q (b_i)_r \prod_{i=1}^k (d_i)_s}{\prod_{i=1}^j (g_i)_{r+s} \prod_{i=1}^m (e_i)_r \prod_{i=1}^n (h_i)_s} \frac{x^r y^s}{r! s!} \quad (1.8.1)$$

where (a_p) abbreviates the array of p parameters given by $a_1, a_2, \dots, a_p$ with similar interpretations for $(b_q), (d_k)$ et cetera and for convergence of double hypergeometric series(1.8.1), we have

- (i) $p + q < j + m + 1, p + k < j + n + 1, |x| < \infty$ and $|y| < \infty$ or
- (ii) $p + q = j + m + 1, p + k = j + n + 1$, and

$$\begin{cases} |x|^{\frac{1}{p-j}} + |y|^{\frac{1}{p-j}} < 1 & \text{if } p > j \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq j \end{cases}$$

The Appell's double hypergeometric functions F_1, F_2, F_3 and F_4 [23, p.224 (5.7.1.6, ..., 5.7.1.9)] are denoted by $F_{1:0;0}^{1:1;1}, F_{0:1;1}^{1:1;1}, F_{1:0;0}^{0:2;2}$ and $F_{0:1;1}^{2:0;0}$ respectively.

1.9 Exton's Double Hypergeometric Functions

In 1979, Exton [39, p.339(13)] defined the following double hypergeometric function

$$\begin{aligned} & \mathcal{H}_{E:G;M;N}^{A:B;C;D} \left[\begin{matrix} (a_A) : (b_B); (c_C); (d_D) ; \\ (e_E) : (g_G); (m_M); (n_N); \end{matrix} ; x, y \right] \\ &= \sum_{i,j=0}^{\infty} \frac{[(a_A)]_{2i+j} [(b_B)]_{i+j} [(c_C)]_i [(d_D)]_j}{[(e_E)]_{2i+j} [(g_G)]_{i+j} [(m_M)]_i [(n_N)]_j} \frac{x^i y^j}{i! j!} \end{aligned} \quad (1.9.1)$$

Making suitable adjustments in the numbers of numerator and denominator parameters of (1.9.1), we obtain Kampé de Fériet double hypergeometric function [118, p.423 (26)] given by $F_{G:M;N}^{B:C;D} = \mathcal{H}_{0:G;M;N}^{0:B;C;D}$, an additional double hypergeometric function of Exton [41, p.137 (1.2)] given by $X_{E:M;N}^{A:C;D} = \mathcal{H}_{E:0;M;N}^{A:0;C;D}$, Appell's

four double hypergeometric functions [23, p.224 (6,7,8,9)] F_1, F_2, F_3, F_4 . Humbert's double hypergeometric functions [23, pp.225-226 (20,21,22,23,24,25,26)]; see also 37; 116] $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$, Horn's non-confluent double hypergeometric functions [23, p.225 (15,16)] H_3 and H_4 and its confluent double hypergeometric functions [23, p.226 (34,35)]; see also 52] $\mathbf{H}_6$ and $\mathbf{H}_7$ respectively.

In 1982, Exton [41, p.137 (1.2)] defined the following double hypergeometric series

$$X_{E:G;H}^{A:B;D} \left[\begin{matrix} (a_A) : (b_B); (d_D); \\ (e_E) : (g_G); (h_H); \end{matrix} \middle| x, y \right] = \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{2m+n} [(b_B)]_m [(d_D)]_n}{[(e_E)]_{2m+n} [(g_G)]_m [(h_H)]_n} \frac{x^m y^n}{m! n!}, \quad (1.9.2)$$

which is the generalization and unification of Horn's non-confluent double hypergeometric function [23, p.225 (16)]; see also 51; 52; 53] H_4 and Horn's confluent double hypergeometric function [23, p.226 (35)]; see also 53] $\mathbf{H}_7$.

In 1984, Exton [42, p.113 (1.2)] defined the double hypergeometric series of second order. The series is quite similar to one obtained by Kampé de Fériet in the year 1921 and is a generalization of Horn's non-confluent double hypergeometric functions G_2, H_2 [23, pp.224-225 (11,14)] and confluent hypergeometric functions $\Gamma_1, \Gamma_2, \mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_4, \mathbf{H}_5$ and $\mathbf{H}_{11}$ [23, pp.226-227 (27,28,30,31,32,33,39)].

$$G_{E:G;H}^{A:B;D} \left[\begin{matrix} (a_A) : (b_B); (d_D); \\ (e_E) : (g_G); (h_H); \end{matrix} \middle| x, y \right] = \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{m-n} [(b_B)]_m [(d_D)]_n}{[(e_E)]_{m-n} [(g_G)]_m [(h_H)]_n} \frac{x^m y^n}{m! n!}, \quad (1.9.3)$$

where (a_A) and $[(a_A)]_{m-n}$ are defined in the same way as in the preceding section with similar interpretations for others.

1.10 Srivastava's General Triple Hypergeometric Functions $F^{(3)}$

Triple hypergeometric function $F^{(3)}$ of Srivastava [103, p.428] is the unification and generalization of fourteen Lauricella's hypergeometric functions [70, pp.113-114, p.150] $F_A^{(3)}, F_B^{(3)}, F_C^{(3)}, F_D^{(3)}$ including Saran's ten triple hypergeometric

functions [91, pp.293-294 (2.1, 2.2, ..., 2.10); see also 60; 88; 89; 90] $F_E, F_F, F_G, F_K, F_M, F_N, F_P, F_R, F_S, F_T$, three additional functions H_A, H_B, H_C of Srivastava [104, pp.99-100; see also 99; 100; 101; 102], Confluent triple hypergeometric functions of Jain [56, p.396] and Exton [28, pp.80-81], extended definition of Saran's function F_K given by Sharma [93, p.613 (2)], Schl fli's function [92], Mayr's function [77, p.265], Manocha's function [73, p.86; see also 117, p.280 (29)] Φ_M , Erd lyi's function [117, p.222 (19); see also 22] $\Phi^{(3)}$, Erd lyi's function [21, p.446 (7.2); see also 22] $\Phi_2^{(3)}$, Humbert's function [54, p.429; see also 119, p.356 (7)] $\Psi_2^{(3)}$, Gupta's function [49, p.169 (3.1); see also 117, p.240 (8); 108, p.401 (12)] Ψ_K , Exton's functions [37, p.43 (2.1.1.4, 2.1.1.5)] $\Phi_3^{(3)}, \Xi_1^{(3)}$, Carlson's function of three variables [15, p.453 (2.1)] R , Srivastava-Exton function [115, p.373 (12,13)] $\Phi_D^{(3)}$, Appell's functions F_1, F_2, F_3, F_4 , Humbert's confluent functions $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$, Kamp  de F riet's function [58] $F_{C:D}^{A:B}$ in the notation of Burchnall and Chaundy [10, p.112], generalized Kamp  de F riet function $F_{C:D;H}^{A:B;G}$ in the slightly modified notation of Srivastava and Panda, Exton's function [37, p.89 (3.4.1, 3.4.2)] ${}_{(1)}^{(K)}E_D^{(3)}, {}_{(2)}^{(K)}E_D^{(3)}$, Chandel's function [16, p.120 (2.3); see also 17] ${}_{(3)}^{(K)}E_C^{(3)}$, generalized hypergeometric function of one variable [98, p.41 (2.1.1.3)] ${}_A F_B$, Gauss ordinary hypergeometric function [45] ${}_2F_1$, K ummer's confluent hypergeometric function [69] ${}_1F_1$, Bessel function [117, p.37 (9)] ${}_0F_1$, Exponential function [117, p.37 (10)] ${}_0F_0$ and Binomial function [117, p.44 (8)] ${}_1F_0$ etc. Triple series $F^{(3)}$ is given by:

$$F^{(3)} \left[\begin{array}{l} (a_A) :: (b_B); (d_D); (e_E) : (g_G); (h_H); (l_L); \\ (m_M) :: (n_N); (p_P); (q_Q) : (r_R); (s_S); (t_T); \end{array} \middle| x, y, z \right]$$

$$= \sum_{i,j,k=0}^{\infty} \frac{[(a_A)]_{i+j+k} [(b_B)]_{i+j} [(d_D)]_{j+k} [(e_E)]_{k+i} [(g_G)]_i [(h_H)]_j [(l_L)]_k}{[(m_M)]_{i+j+k} [(n_N)]_{i+j} [(p_P)]_{j+k} [(q_Q)]_{k+i} [(r_R)]_i [(s_S)]_j [(t_T)]_k} \frac{x^i y^j z^k}{i! j! k!}, \quad (1.10.1)$$

where (a_A) represents the array of A parameters given by $a_1 a_2 \dots a_A$ and $[(a_A)]_{i+j+k}$ represents the $(a_1)_{i+j+k} (a_2)_{i+j+k} \dots (a_A)_{i+j+k}$ with similar interpretations for others.

The triple hypergeometric series $F^{(3)}$ is convergent [66, pp.32-33; 18, p.156; 20,

p.40; 117, p.70 (41)], when

$$\begin{cases} M + N + Q + R > A + B + E + G \\ M + N + P + S > A + B + D + H \\ M + P + Q + T > A + D + E + L \end{cases}$$

or equivalently

$$\begin{cases} 1 + M + N + Q + R - A - B - E - G > 0 \\ 1 + M + N + P + S - A - B - D - H > 0 \\ 1 + M + P + Q + T - A - D - E - L > 0 \end{cases}$$

where A, B, D, E , etc. are the non-negative integers and $|x| < 1$, $|y| < 1$ and $|z| < 1$; but in case

$$\begin{cases} 1 + M + N + Q + R = A + B + E + G \\ 1 + M + N + P + S = A + B + D + H \\ 1 + M + P + Q + T = A + D + E + L \end{cases}$$

then $|x|$, $|y|$ and $|z|$ are to be restricted in such a way that the series involved, are either terminating or convergent.

1.11 Multi-Variable Extension of Kampé de Fériet's Functions

The multi-variable extension of Kampé de Fériet function is given in the form [113; see also 117, pp. 65-66]

$$\begin{aligned} & F_{\ell; m_1; m_2; \dots; m_n}^{p; q_1; q_2; \dots; q_n} \left[\begin{matrix} (a_p) : (b_{q_1}^{(1)}); \dots; (b_{q_n}^{(n)}); \\ (\alpha_\ell) : (\beta_{m_1}^{(1)}); \dots; (\beta_{m_n}^{(n)}); \end{matrix} \middle| x_1, \dots, x_n \right] \\ &= \sum_{s_1, \dots, s_n=0}^{\infty} \Lambda(s_1, \dots, s_n) \frac{x_1^{s_1}}{s_1!} \dots \frac{x_n^{s_n}}{s_n!} \end{aligned} \quad (1.11.1)$$

$$\Lambda(s_1, \dots, s_n) = \frac{\prod_{j=1}^p (a_j)_{s_1+\dots+s_n} \prod_{j=1}^{q_1} (b_j^{(1)})_{s_1} \cdots \prod_{j=1}^{q_n} (b_j^{(n)})_{s_n}}{\prod_{j=1}^{\ell} (\alpha_j)_{s_1+\dots+s_n} \prod_{j=1}^{m_1} (\beta_j^{(1)})_{s_1} \cdots \prod_{j=1}^{m_n} (\beta_j^{(n)})_{s_n}}$$

and, for convergence of the multiple hypergeometric series in (1.11.1),

$$1 + \ell + m_k - p - q_k \geq 0, \quad k = 1, \dots, n;$$

the equality holds when, in addition, either

$$p > \ell \quad \text{and} \quad |x_1|^{\frac{1}{p-\ell}} + \dots + |x_n|^{\frac{1}{p-\ell}} < 1$$

or

$$p \leq \ell \quad \text{and} \quad \max\{|x_1|, \dots, |x_n|\} < 1$$

1.12 Srivastava-Daoust Multi-Variable Hypergeometric Functions

A further generalization of the Kampé de Fériet function is due to Srivastava and Daoust who indeed, defined an extension of Wright's ${}_p\psi_q$ function in two variables. More generally recalling here the following extension of the Wright's function ${}_p\psi_q$ in *several* variables, which is referred in the literature as the generalized Lauricella function of several variables, it is also due to Srivastava and Daoust [113, p.454].

$$\begin{aligned} F_{C : D^{(1)} ; \dots ; D^{(n)}}^A : B^{(1)} ; \dots ; B^{(n)} \left(\begin{array}{c} [(a) : \theta^{(1)}, \dots, \theta^{(n)}] : [(b^{(1)}) : \phi^{(1)}] ; \dots ; [(b^{(n)}) : \phi^{(n)}] ; \\ \\ [(c) : \psi^{(1)}, \dots, \psi^{(n)}] : [(d^{(1)}) : \delta^{(1)}] ; \dots ; [(d^{(n)}) : \delta^{(n)}] ; \end{array} \right. z_1, \dots, z_n \Bigg) \\ = \sum_{m_1, \dots, m_n=0}^{\infty} \Omega(m_1, \dots, m_n) \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!} \end{aligned} \quad (1.12.1)$$

where, for convenience,

$$\Omega(m_1, \dots, m_n) = \frac{\prod_{j=1}^A (a_j)_{m_1\theta_j^{(1)}+\dots+m_n\theta_j^{(n)}} \prod_{j=1}^{B^{(1)}} (b_j^{(1)})_{m_1\phi_j^{(1)}} \cdots \prod_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^C (c_j)_{m_1\psi_j^{(1)}+\dots+m_n\psi_j^{(n)}} \prod_{j=1}^{D^{(1)}} (d_j^{(1)})_{m_1\delta_j^{(1)}} \cdots \prod_{j=1}^{D^{(n)}} (d_j^{(n)})_{m_n\delta_j^{(n)}}} \quad (1.12.2)$$

the coefficients

$$\begin{cases} \theta_j^{(k)}, j = 1, \dots, A; \phi_j^{(k)}, j = 1, \dots, B^{(k)}; \psi_j^{(k)}, j = 1, \dots, C; \\ \delta_j^{(k)}, j = 1, \dots, D^{(k)}; \forall k \in \{1, \dots, n\} \end{cases} \quad (1.12.3)$$

are real and positive, and (a) abbreviates the array of A parameters $a_1, \dots, a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}, j = 1, \dots, B^{(k)}; \forall k \in \{1, \dots, n\}$$

with similar interpretations for (c) and $(d^{(k)})$, $k = 1, \dots, n$; *et cetera*. For the precise conditions under which the multiple hypergeometric series in (1.12.1) converges absolutely [see 114, pp. 157 – 158; see also 50].

1.13 Transformation and Reduction Formulae

For transformations and reduction formulae of multiple hypergeometric functions and product theorems of hypergeometric functions, we refer the monographs and research papers [1, 2, 23, 85, 98, 117].

Pfaff Kummer Linear Transformation:

$${}_2F_1 \left[\begin{matrix} a, b; \\ c; \end{matrix} t \right] = (1-t)^{(-a)} {}_2F_1 \left[\begin{matrix} a, c-b; \\ c; \end{matrix} \frac{-t}{1-t} \right] \quad (1.13.1)$$

$$c \neq 0, -1, -2, \dots, \quad |\arg(1-t)| < \pi$$

Euler's Linear Transformation:

If $|z| < 1$; $|\frac{z}{1-z}| < 1$; $|\arg(1-z)| < \pi$, $c \neq 0, -1, -2, -3, \dots$, then

$${}_2F_1 \left[\begin{matrix} a, b; \\ c; \end{matrix} z \right] = (1-z)^{c-a-b} {}_2F_1 \left[\begin{matrix} c-a, c-b; \\ c; \end{matrix} z \right] \quad (1.13.2)$$

Kummer's I Transformation:

$${}_1F_1 \left[\begin{matrix} a; \\ c; \end{matrix} t \right] = e^t {}_1F_1 \left[\begin{matrix} c-a; \\ c; \end{matrix} -t \right] \quad (1.13.3)$$

where $c \neq 0, -1, -2, \dots$

Kummer's II Transformation:

$${}_1F_1 \left[\begin{matrix} a; \\ 2a; \end{matrix} 2t \right] = e^t {}_0F_1 \left[\begin{matrix} -; \\ a + \frac{1}{2}; \end{matrix} \frac{t^2}{4} \right] \quad (1.13.4)$$

where $2a$ is not an odd integer < 0 i.e. $2a \neq -1, -3, -5, \dots$

Whipple Quadratic Transformation: [1, p.130, (3.1.15)]

$${}_3F_2 \left[\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix} ; t \right] = (1-t)^{-a} {}_3F_2 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, 1+a-b-c \\ 1+a-b, 1+a-c \end{matrix} ; -\frac{4t}{(1-t)^2} \right] \quad (1.13.5)$$

Some other results:

$$\left\{ {}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \alpha + \beta + \frac{1}{2} \end{matrix} ; z \right] \right\}^2 = {}_3F_2 \left[\begin{matrix} 2\alpha, 2\beta, \alpha + \beta \\ 2\alpha + 2\beta, \alpha + \beta + \frac{1}{2} \end{matrix} ; z \right] \quad (1.13.6)$$

$${}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \alpha + \beta - \frac{1}{2} \end{matrix} ; z \right] {}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \alpha + \beta + \frac{1}{2} \end{matrix} ; z \right] = {}_3F_2 \left[\begin{matrix} 2\alpha, 2\beta, \alpha + \beta \\ 2\alpha + 2\beta - 1, \alpha + \beta + \frac{1}{2} \end{matrix} ; z \right] \quad (1.13.7)$$

$${}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \alpha + \beta - \frac{1}{2} \end{matrix} ; z \right] {}_2F_1 \left[\begin{matrix} \alpha, \beta - 1 \\ \alpha + \beta - \frac{1}{2} \end{matrix} ; z \right] = {}_3F_2 \left[\begin{matrix} 2\alpha, 2\beta - 1, \alpha + \beta - 1 \\ 2\alpha + 2\beta - 2, \alpha + \beta - \frac{1}{2} \end{matrix} ; z \right] \quad (1.13.8)$$

$${}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \alpha + \beta + \frac{1}{2} \end{matrix} ; z \right] {}_2F_1 \left[\begin{matrix} \frac{1}{2} - \alpha, \frac{1}{2} - \beta \\ \frac{3}{2} - \alpha - \beta \end{matrix} ; z \right] = {}_3F_2 \left[\begin{matrix} \alpha - \beta + \frac{1}{2}, \beta - \alpha + \frac{1}{2}, \frac{1}{2} \\ \alpha + \beta + \frac{1}{2}, \frac{3}{2} - \alpha - \beta \end{matrix} ; z \right] \quad (1.13.9)$$

$${}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \alpha + \beta + \frac{1}{2} \end{matrix} ; z \right] {}_2F_1 \left[\begin{matrix} \alpha + \frac{1}{2}, \beta + \frac{1}{2} \\ \alpha + \beta + \frac{1}{2} \end{matrix} ; z \right] = (1-z)^{-\frac{1}{2}} {}_3F_2 \left[\begin{matrix} 2\alpha, 2\beta, \alpha + \beta \\ 2\alpha + 2\beta, \alpha + \beta + \frac{1}{2} \end{matrix} ; z \right] \quad (1.13.10)$$

1.14 Multiple Series Identities

Series Rearrangement Technique:

The technique described and illustrated in the thesis is based, in part, upon certain interchanges of the order of a double (or multiple) summation. Such

rearrangement of terms in iterated finite series can be justified in the elementary sense when, for example, the series involved are absolutely convergent. Thus the identities contained in lemmas 1, 2, 3, 4, 5 and 6 below may be considered as purely formal.

LEMMA 1

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n-k) \quad (1.14.1)$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^n B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n+k) \quad (1.14.2)$$

LEMMA 2

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} A(k, n-2k) \quad (1.14.3)$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n+2k) \quad (1.14.4)$$

where, and in what follows, $[x]$ denotes the greatest integer in x .

LEMMA 3 For any positive integer m ,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} A(k, n-mk) \quad (1.14.5)$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n+mk) \quad (1.14.6)$$

For $m = 1$, Lemma 3 evidently reduces to Lemma 1, while a special case of Lemma 3 when $m = 2$ is precisely Lemma 2. These special cases of Lemma 3 were stated and proved by [85, p.56 sec. 37].

LEMMA 4 Multiple Series Identities: [117, p.102 (16);(17)]

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \phi(n; k_1, k_2, \dots, k_m) \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{\substack{k_1+k_2+\dots+k_m \leq n \\ k_1, k_2, \dots, k_m=0}} \phi(n - k_1 - k_2 - \dots - k_m; k_1, k_2, \dots, k_m) \right) \\ & \quad \sum_{n=0}^{\infty} \left(\sum_{\substack{k_1+k_2+\dots+k_m \leq n \\ k_1, k_2, \dots, k_m=0}} \phi(n; k_1, k_2, \dots, k_m) \right) \end{aligned} \quad (1.14.7)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_m=0}^{\infty} \phi(n + k_1 + k_2 + \dots + k_m; k_1, k_2, \dots, k_m) \right) \quad (1.14.8)$$

LEMMA 5 Finite Double Series Identity: [117, p.103, (2.1.18)]

$$\sum_{n=0}^p \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} B(k, n) = \sum_{n=0}^p \sum_{k=0}^{\lfloor \frac{p-n}{m} \rfloor} B(k, n + mk) \quad (1.14.9)$$

where, and in what follows, $\lfloor x \rfloor$ denotes the greatest integer in x .

$$\sum_{r=0}^m \sum_{s=0}^r B(s, r) = \sum_{r=0}^m \sum_{s=0}^{m-r} B(s, s + r) \quad (1.14.10)$$

LEMMA 6 Series Identity: [117, p.214 Q.8]

$$\sum_{m,n=0}^{\infty} \Psi(m, n) = \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} \sum_{m,n=0}^{\infty} \Psi(mM + j, nN + k) \quad (1.14.11)$$

where $\forall M, N \in \{1, 2, 3, \dots\}$

$$\sum_{m=0}^{\infty} \sum_{q=0}^{\infty} A(m, q) = \sum_{j=0}^{2n-1} \sum_{i=0}^{2-1} \sum_{m,q=0}^{\infty} A(2nm + j, 2q + i) \quad (1.14.12)$$

Decomposition Technique:

$$\sum_{m=0}^{\infty} \Phi(m) = \sum_{m=0}^{\infty} \Phi(2m) + \sum_{m=0}^{\infty} \Phi(2m + 1) \quad (1.14.13)$$

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(m, n) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(2m, 2n) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(2m, 2n + 1) \\ &+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(2m + 1, 2n) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Psi(2m + 1, 2n + 1) \end{aligned} \quad (1.14.14)$$

Few More Summation Properties:

$$m > n, \quad \sum_{k=0}^n (-n)_k (-k)_m \phi(k) = 0 \quad (1.14.15)$$

$$n > m, \quad \sum_{k=0}^n (-n)_k (-k)_m \phi(k) = \sum_{k=0}^{n-m} (-n)_{k+m} (-k - m)_m \phi(k + m) \quad (1.14.16)$$

$$n = m, \quad \sum_{k=0}^n (-n)_k (-k)_n \phi(k) = [(-1)^n n!]^2 \phi(n) \quad (1.14.17)$$

$$\sum_{r=0}^n \phi(r) = \sum_{r=0}^n \phi(n-r) \quad (1.14.18)$$

For a positive integer m , [117, p.161]

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} F(k, n-mk) = \sum_{k=0}^{\infty} \sum_{n=0}^{\lfloor \frac{k}{m} \rfloor} F(k-mn, n) \quad (1.14.19)$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} F(k, n-mk+k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\lfloor \frac{k}{m} \rfloor} F(k-mn, n-mn+k) \quad (1.14.20)$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{m} \rfloor} F(k, n) = \sum_{k=0}^{\infty} \sum_{n=0}^{\lfloor \frac{k}{m} \rfloor} F(k-mn, n-m^2n+mk) \quad (1.14.21)$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} F(k, n-rk) = \sum_{k=0}^{\infty} \sum_{n=0}^{\lfloor \frac{k}{q} \rfloor} F(k-qn, n+(p-r)(k-qn)) \quad (1.14.22)$$

where p, q and r are integers, $p \geq 1$, $q \geq 1$ and $r \geq 0$.

1.15 Some Hypergeometric Polynomials, their Relationships and Generating Relations

1.15.1 Generalized Laguerre Polynomials of One Variable

Generalized Laguerre polynomials of one variable $L_n^{(\alpha)}(x)$ possess the following generating relation [85, p.130]

$$e^t {}_0F_1(-; 1+\alpha; -xt) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)t^n}{(1+\alpha)_n} \quad (1.15.1)$$

Laguerre Polynomial [117, p.131 (2.5.1)]

$$L_n^{(\alpha)}(x) = \lim_{|\beta| \rightarrow \infty} \left\{ P_n^{(\alpha, \beta)} \left(1 - \frac{2x}{\beta} \right) \right\} \quad (1.15.2)$$

A generating relation [27, p.249 (19.7.1)] is given by

$$(t-1)^m e^{xt} = \sum_{n=-m}^{\infty} x^n L_m^{(n)}(x) m! \frac{t^{m+n}}{(m+n)!} \quad (1.15.3)$$

Generalization of the generating function [117, p.250] is given by

$$\sum_{n=0}^{\infty} L_m^{(n-m)}(x) \frac{t^n}{n!} = \frac{(t-x)^m}{m!} e^t \quad (1.15.4)$$

1.15.2 Classical Jacobi Polynomial of One Variable

The Jacobi's polynomials $P_n^{(\alpha,\beta)}(x)$ [85, p.254 (1)] are given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1+\alpha+\beta+n; \\ 1+\alpha \end{matrix} ; \frac{1-x}{2} \right] = (-1)^n P_n^{(\beta,\alpha)}(-x) \quad (1.15.5)$$

The Laguerre's polynomials $L_n^{(\alpha)}(x)$ [85, p.200 (1)] are given by

$$\lim_{|\beta| \rightarrow \infty} P_n^{(\alpha,\beta)} \left(1 - \frac{2x}{\beta} \right) = L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1 \left[\begin{matrix} -n \\ 1+\alpha \end{matrix} ; x \right] \quad (1.15.6)$$

A transformation for Jacobi polynomial [121, p. 64 (4.22.1)] is given by

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{1-x}{2} \right)^n P_n^{(-\alpha-\beta-2n-1,\beta)} \left(\frac{x+3}{x-1} \right) \quad (1.15.7)$$

$$P_n^{(\alpha,\beta-n)}(x) = \left(\frac{1-x}{2} \right)^n P_n^{(-\alpha-\beta-n-1,\beta-n)} \left(\frac{x+3}{x-1} \right) \quad (1.15.8)$$

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{1+x}{2} \right)^n P_n^{(\alpha,-\alpha-\beta-2n-1)} \left(\frac{3-x}{x+1} \right) \quad (1.15.9)$$

1.15.3 Chebyshev (Tchebycheff) Polynomials of I and II

Kind [24, p.186(24), (25)]

$$T_n(t) = \cos[n\{\cos^{-1}(t)\}] = {}_2F_1 \left[\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix} ; \frac{1-t}{2} \right] \quad (1.15.10)$$

$$U_n(t) = \frac{\sin[(n+1)\{\cos^{-1}(t)\}]}{\sqrt{(1-t^2)}} = (n+1) {}_2F_1 \left[\begin{matrix} -n, n+2 \\ \frac{3}{2} \end{matrix} ; \frac{1-t}{2} \right] \quad (1.15.11)$$

Tchebycheff Polynomials of I and II Kind [85, p.301 (1,2,3,4); see also 117, p.36]

is given by

$$T_n(x) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) \quad (1.15.12)$$

$$U_n(x) = \frac{(n+1)!}{\left(\frac{3}{2}\right)_n} P_n^{\left(\frac{1}{2}, \frac{1}{2}\right)}(x) \quad (1.15.13)$$

$$(1 - 2xt + t^2)^{-1} = \sum_{n=0}^{\infty} U_n(x) t^n \quad (1.15.14)$$

$$\sum_{n=0}^{\infty} T_n(x) t^n = (1 - xt)(1 - 2xt + t^2)^{-1} \quad (1.15.15)$$

$$\sum_{n=0}^{\infty} U_n(x) t^n = (1 - 2xt + t^2)^{-1} \quad (1.15.16)$$

1.15.4 Gegenbauer Polynomials and their Special Case

$$\begin{aligned} C_n^\nu(z) &= \frac{\binom{n+2\nu-1}{n}}{\binom{n+\nu-\frac{1}{2}}{n}} P_n^{\left(\nu-\frac{1}{2}, \nu-\frac{1}{2}\right)}(z) \\ &= \frac{\Gamma(n+2\nu)}{n! \Gamma(2\nu)} \frac{n! \Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(n+\nu+\frac{1}{2}\right)} P_n^{\left(\nu-\frac{1}{2}, \nu-\frac{1}{2}\right)}(z) \\ &= \frac{(2\nu)_n}{\left(\nu+\frac{1}{2}\right)_n} P_n^{\left(\nu-\frac{1}{2}, \nu-\frac{1}{2}\right)}(z) \end{aligned} \quad (1.15.17)$$

Gegenbauer Polynomial [117, p.137 (5)] is defined as

$$C_n^\alpha(x) = \frac{(2\alpha)_n}{n!} x^n {}_2F_1 \left[\begin{matrix} \Delta(2; -n); & x^2 - 1 \\ \alpha + \frac{1}{2} & ; & x^2 \end{matrix} \right] \quad (1.15.18)$$

Gegenbauer Polynomial [85, p.280 (19)]

$$C_n^{\frac{1}{2}}(x) = P_n(x) = \frac{\left(\frac{1}{2}\right)_n (2x)^n}{n!} {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{-n+1}{2}; & 1 \\ \frac{1}{2} - n & ; & x^2 \end{matrix} \right] \quad (1.15.19)$$

$$C_n^\nu(x) = \frac{(\nu)_n (2x)^n}{n!} {}_2F_1 \left[\begin{matrix} -\frac{n}{2}, \frac{-n+1}{2}; & 1 \\ 1 - \nu - n; & x^2 \end{matrix} \right] \quad (1.15.20)$$

$$(1 - 2xt + t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^\nu(x) t^n \quad (1.15.21)$$

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n \quad (1.15.22)$$

Gegenbauer or Ultraspherical polynomials [121, p.81 (4.7.1)]

$$P_n^{(\nu)}(x) = \left[\frac{(2\nu)_n}{\left(\nu+\frac{1}{2}\right)_n} \right] P_n^{\left(\nu-\frac{1}{2}, \nu-\frac{1}{2}\right)}(x) = C_n^\nu(x); \quad \nu > -\frac{1}{2} \quad (1.15.23)$$

$$P_n^{(\alpha, \alpha)}(x) = \binom{\alpha+n}{n} \binom{2\alpha+n}{n}^{-1} C_n^{\alpha+\frac{1}{2}}(x) \quad (1.15.24)$$

$$P_n^{(0,0)}(x) = C_n^{\frac{1}{2}}(x) = P_n^{(\frac{1}{2})}(x) = P_n(x) \quad (1.15.25)$$

$$U_n(x) = C_n^1(x) = P_n^{(1)}(x) \quad (1.15.26)$$

A generating relation for Gegenbauer polynomial [85, p.278 (7)] is given by

$$\begin{aligned} e^{xt} {}_0F_1 \left[\begin{matrix} - & ; & t^2(x^2 - 1) \\ \nu + \frac{1}{2} & ; & 4 \end{matrix} \right] &= {}_0F_0 \left[\begin{matrix} - & ; & xt \\ - & ; & \end{matrix} \right] {}_0F_1 \left[\begin{matrix} - & ; & t^2(x^2 - 1) \\ \nu + \frac{1}{2} & ; & 4 \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \frac{C_n^{\nu}(x)t^n}{(2\nu)_n} \end{aligned} \quad (1.15.27)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} C_n^{\alpha}(x)t^n = F_1 \left[\lambda; \alpha, \alpha; \mu; (x + \sqrt{x^2 - 1})t, (x - \sqrt{x^2 - 1})t \right] \quad (1.15.28)$$

1.15.5 Rice and Khandekar Polynomials of One Variable

In the endeavor of attempting to unify several results in the theory of polynomials, also in hypergeometric functions of two or more variables, Khandekar [67, p.158 (2.3)] defined the generalized Rice polynomials of one variable in the following form:-

$$H_n^{(\alpha, \beta)}[\xi, p, \nu] = \frac{(1 + \alpha)_n}{n!} {}_3F_2 \left[\begin{matrix} -n, 1 + \alpha + \beta + n, \xi & ; & \\ & & \nu \\ 1 + \alpha, p & ; & \end{matrix} \right] \quad (1.15.29)$$

where $n = 0, 1, 2, 3, \dots$, and $p \neq -n - 1, -n - 2, -n - 3, \dots$

If $\alpha = \beta = 0$ in (1.15.29), we get Rice polynomials [86, p.108]

$$H_n[\xi, p, \nu] = H_n^{(0,0)}[\xi, p, \nu] \quad (1.15.30)$$

If $p = \xi$ and $\nu = \frac{1}{2}(1 - x)$, (1.15.29) becomes:-

$$H_n^{(\alpha, \beta)} \left[\xi, \xi, \frac{1}{2}(1 - x) \right] = P_n^{(\alpha, \beta)}(x) \quad (1.15.31)$$

or

$$H_n^{(\alpha, \beta)}[\xi, \xi, x] = P_n^{(\alpha, \beta)}(1 - 2x) \quad (1.15.32)$$

where $P_n^{(\alpha, \beta)}(x)$ are the Jacobi polynomials.

1.15.6 Krall and Frink Bessel-Polynomials:

Krall and Frink Bessel's polynomial is defined as

$$y_n(x, \alpha - n, \beta) = n! \left(\frac{-x}{\beta} \right)^n L_n^{(1-\alpha-n)} \left(\frac{\beta}{x} \right) \quad (1.15.33)$$

Krall and Frink [117, p.170 Q.19(ii)] generating relation is given by

$$\sum_{n=0}^{\infty} y_n(x, a - n, b) \frac{t^n}{n!} = \left[1 - \frac{xt}{b} \right]^{1-a} e^t \quad (1.15.34)$$

1.15.7 Rainville's Polynomials:

Rainville polynomials [85] are defined as

$$g_n(x, a, b) = (2x)^n {}_3F_1 \left[\begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}, 1+a; \\ 1+b \end{matrix}; -\frac{1}{x^2} \right] \quad (1.15.35)$$

$$\sum_{n=0}^{\infty} g_n(x, a, b) \frac{t^n}{n!} = e^{2xt} {}_1F_1 \left[\begin{matrix} 1+a; \\ 1+b \end{matrix}; -t^2 \right] \quad (1.15.36)$$

When $b = a$, we get

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \quad (1.15.37)$$

A polynomial and generating function [85, p.298 (1),(4)] is given by

$$R_n(a, x) = \frac{(a)_{2n}}{n!(a)_n} {}_1F_1(-n; a+n; x), \quad (1.15.38)$$

which are related to the proper simple Laguerre polynomial

$$L_n(x) = {}_1F_1(-n; 1; x) \quad (1.15.39)$$

$$e^{2t} {}_0F_1 \left(-; \frac{1}{2} + \frac{a}{2}; t^2 - xt \right) = \sum_{n=0}^{\infty} \frac{R_n(a, x) t^n}{\left(\frac{1}{2} + \frac{a}{2} \right)_n} \quad (1.15.40)$$

1.16 Types of Generating Relations

Linear Generating Functions:

Consider a two-variable function $F(x, t)$ which possesses a formal (not necessarily convergent for $t \neq 0$) power series expansion in t such that

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x) t^n$$

where each member of the coefficient set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of t . Then the expansion of $F(x, t)$ is said to have generated the set $f_n(x)$ and $F(x, t)$ is called a linear generating function (or, simply, a generating function) for the set $f_n(x)$.

The definition may be extended slightly to include a generating function of the type

$$G(x, t) = \sum_{n=0}^{\infty} c_n g_n(x) t^n \quad (1.16.1)$$

where the sequence $\{c_n\}_{n=0}^{\infty}$ may contain the parameters of the set $g_n(x)$, but is independent of x and t . A set of functions may have more than one generating function. However, if $G(x, t) = \sum_{n=0}^{\infty} h_n(x) t^n$ then $G(x, t)$ is the unique generator for the set $h_n(x)$ as the coefficient set.

Now we, extend our definition of a generating relation to include functions which possess Laurent series expansion. Thus, if the set $f_n(x)$ is defined for $n = 0, \pm 1, \pm 2, \dots$, the definition (1.16.1) may be extended in terms of Laurent series expansion

$$F^*(x, t) = \sum_{n=-\infty}^{\infty} \gamma_n f_n(x) t^n$$

where the sequence $\{\gamma_n(x)\}_{n=0}^{\infty}$ is independent of x and t .

Bilinear Generating Functions:

If a three-variable function $F(x, y, t)$ possesses a formal power series expansion in t such that

$$F(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n,$$

where the sequence $\{\gamma_n\}$ is independent of x, y and t , then $F(x, y, t)$ is called a Bilinear generating function for the set $\{f_n(x)\}$.

More generally, if $\mathcal{F}(x, y, t)$ can be expanded in powers of t in the form

$$\mathcal{F}(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) f_{\beta(n)}(y) t^n,$$

where $\alpha(n)$ and $\beta(n)$ are functions of n which are not necessarily equal, we shall still call $\mathcal{F}(x, y, t)$ a bilinear generating function for the set $\{f_n(x)\}$.

Bilateral Generating Functions:

Suppose that a three-variable function $H(x, y, t)$ has a formal power series expansion in t such that

$$H(x, y, t) = \sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n,$$

where the sequence $\{h_n\}_{n=0}^{\infty}$ is independent of x, y and t , and the sets of functions $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(y)\}_{n=0}^{\infty}$ are different. Then $H(x, y, t)$ is called a bilateral generating function for the set $\{f_n(x)\}$ or $\{g_n(y)\}$. The above definition of a bilateral generating function, used earlier by Rainville [85, p.170] and McBride [78, p.19], may be extended to include bilateral generating functions of the type

$$\mathcal{H}(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) g_{\beta(n)}(y) t^n,$$

where the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is independent of x, y and t , the sets of functions $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(y)\}_{n=0}^{\infty}$ are different, and $\alpha(n)$ and $\beta(n)$ are functions of n which are not necessarily equal.

Extended Linear Generating Functions:

An extended linear generating function is a generating function of the type

$$F(m; x, t) = \sum_{n=0}^{\infty} \lambda_{m,n} f_{m+n}(x) t^n$$

where m is a fixed integer ≥ 0 .

1.17 Few More Results:

$${}_2F_1 \left[\begin{matrix} b, b - \frac{1}{2} \\ 2b \end{matrix} ; x \right] = \left(\frac{2}{1 + \sqrt{1-x}} \right)^{2b-1} \quad (1.17.1)$$

$${}_2F_1 \left[\begin{matrix} b, b + \frac{1}{2} \\ 2b \end{matrix} ; x \right] = (1-x)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{1-x}} \right)^{2b-1} \quad (1.17.2)$$

$${}_2F_1 \left[\begin{matrix} b, b + \frac{1}{2} \\ \frac{1}{2} \end{matrix} ; x^2 \right] = \frac{(1+x)^{-2b} + (1-x)^{-2b}}{2} \quad (1.17.3)$$

$${}_2F_1 \left[\begin{matrix} b, b + \frac{1}{2} ; \\ \frac{3}{2} ; \end{matrix} x^2 \right] = \frac{(1+x)^{1-2b} - (1-x)^{1-2b}}{2x(1-2b)} \quad (1.17.4)$$

$${}_2F_1 \left[\begin{matrix} b, -b ; \\ \frac{1}{2} ; \end{matrix} -x \right] = \frac{1}{2} \left[(\sqrt{(1+x)} + \sqrt{x})^{2b} + (\sqrt{(1+x)} - \sqrt{x})^{2b} \right] \quad (1.17.5)$$

$${}_2F_1 \left[\begin{matrix} b, 1-b ; \\ \frac{1}{2} ; \end{matrix} -x \right] = \frac{1}{2\sqrt{(1+x)}} \left[(\sqrt{(1+x)} + \sqrt{x})^{2b-1} + (\sqrt{(1+x)} - \sqrt{x})^{2b-1} \right] \quad (1.17.6)$$

putting $x = \frac{t^2}{4}$, we find an important result used in the thesis:

$${}_2F_1 \left[\begin{matrix} b, 1-b ; \\ \frac{1}{2} ; \end{matrix} \frac{-t^2}{4} \right] = \frac{1}{2\sqrt{(1+\frac{t^2}{4})}} \cdot \left[\left\{ \sqrt{\left(1+\frac{t^2}{4}\right)} + \frac{t}{2} \right\}^{2b-1} + \left\{ \sqrt{\left(1+\frac{t^2}{4}\right)} - \frac{t}{2} \right\}^{2b-1} \right] \quad (1.17.7)$$

$${}_2F_1 \left[\begin{matrix} b, 1-b ; \\ \frac{3}{2} ; \end{matrix} -x \right] = \frac{1}{2(2b-1)\sqrt{(1+x)}} \left[(\sqrt{(1+x)} + \sqrt{x})^{2b-1} - (\sqrt{(1+x)} - \sqrt{x})^{2b-1} \right] \quad (1.17.8)$$

$${}_2F_1 \left[\begin{matrix} b, 2-b ; \\ \frac{3}{2} ; \end{matrix} -x \right] = \frac{1}{4(b-1)\sqrt{x(1+x)}} \left[(\sqrt{(1+x)} + \sqrt{x})^{2b-2} - (\sqrt{(1+x)} - \sqrt{x})^{2b-2} \right] \quad (1.17.9)$$

Putting $x = \frac{t^2}{4}$, we get

$${}_2F_1 \left[\begin{matrix} b + \frac{1}{2}, 2 - (b + 1/2) ; \\ \frac{3}{2} ; \end{matrix} \frac{-t^2}{4} \right] = \frac{1}{4(b + 1/2 - 1)\sqrt{\frac{t^2}{4}(1+\frac{t^2}{4})}} \cdot \left[\left\{ \sqrt{\left(1+\frac{t^2}{4}\right)} + \frac{t}{2} \right\}^{2b+1-2} - \left\{ \sqrt{\left(1+\frac{t^2}{4}\right)} - \frac{t}{2} \right\}^{2b-1} \right] \quad (1.17.10)$$

$$\begin{aligned} & {}_2F_1 \left[\begin{matrix} b, -b ; \\ \frac{1}{2} ; \end{matrix} \frac{-t^2}{4} \right] + bt {}_2F_1 \left[\begin{matrix} b + \frac{1}{2}, 1 - (b + \frac{1}{2}) ; \\ \frac{3}{2} ; \end{matrix} \frac{-t^2}{4} \right] \\ &= \left(\frac{t + \sqrt{(4+t^2)}}{2} \right)^{2b} = \sum_{n=0}^{\infty} \frac{(b)_{\frac{n}{2}} t^n}{(1+b)_{\frac{-n}{2}} n!} \end{aligned} \quad (1.17.11)$$

An Integral [85, p.31, Q.7]

$$\int_{-1}^{+1} (1+x)^{p-1} (1-x)^{q-1} dx = 2^{p+q-1} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (1.17.12)$$

[117, p.86, Q.1]

If $Re(\alpha) > 0$ and $Re(\beta) > 0$, then

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} dt = (b-a)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}; b \neq a \quad (1.17.13)$$

and

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} (z+t)^{-\alpha-\beta} dt = \frac{B(\alpha, \beta)}{(1+z)^\alpha z^\beta} \quad (1.17.14)$$

; provided $|arg(z+t)| < \pi - \epsilon, 0 < \epsilon < \pi$, for $0 \leq t \leq 1$.

Gauss Theorem:

$${}_2F_1 \left[\begin{matrix} -n, b; \\ c \end{matrix} ; 1 \right] = \frac{(c-b)_n}{(c)_n} \quad (1.17.15)$$

Chapter 2

Certain Hypergeometric Generating Relations via Gould's Identity (Part-I)

2.1 Introduction and Preliminaries

The findings unveiled in the present chapter are the hypergeometric generating relations associated with two hypergeometric polynomials of one variable $H_n^{(\alpha,\beta)}(x;m)$ and $\mathcal{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$ with their independent demonstrations via Gould's identity. As applications, some well known and new generating relations are deduced.

Throughout in the thesis, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \dots\}, \mathbb{N}_0 := \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}, \mathbb{Z}_0^- := \{0, -1, -2, -3, \dots\}, \\ \mathbb{Z}^- := \{-1, -2, -3, \dots\} = \mathbb{Z}_0^- \setminus \{0\} \text{ and } \mathbb{Z} = (\mathbb{Z}_0^- \cup \mathbb{N}).$$

Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \dots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases} \quad (2.1.1)$$

it is being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Some useful consequences of Lagrange's expansion [123, p.133; see also 82, p.146, problem 207] include the following generalization [82, p.349, problem 216] of the familiar binomial expansion :

$$\sum_{n=0}^{\infty} \binom{\theta + (\beta + 1)n}{n} t^n = \frac{(1 + \zeta)^{\theta+1}}{(1 - \beta\zeta)} \quad (2.1.2)$$

where $\binom{\theta + (\beta + 1)n}{n}$ is a binomial coefficient and θ, β are complex numbers independent of n and ζ is a function of 't' defined implicitly by

$$\zeta = t(1 + \zeta)^{1+\beta} \quad (2.1.3)$$

subject to the condition

$$\zeta(0) = 0 \quad (2.1.4)$$

Another generalization [82, p.348, problem 212] related with the equation (2.1.2), is given as:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\theta}{\{\theta + (\beta + 1)n\}} \binom{\theta + (\beta + 1)n}{n} t^n \\ &= 1 + \theta \sum_{n=1}^{\infty} \binom{\theta + (\beta + 1)n - 1}{n-1} \frac{t^n}{n} = (1 + \zeta)^\theta \end{aligned} \quad (2.1.5)$$

where ζ is defined by the equations (2.1.3) and (2.1.4).

When $\beta = -1$, both results (2.1.2) and (2.1.5) reduce immediately to the binomial expansion.

Gould [46, p.90; see also 87, p.169] gave the following identity:

$$\sum_{n=0}^{\infty} \frac{\theta(\sigma + \mu n)}{\{\theta + (\beta + 1)n\}} \binom{\theta + (\beta + 1)n}{n} t^n = (1 + \zeta)^\theta \left(\sigma + \frac{\mu\theta\zeta}{1 - \beta\zeta} \right) \quad (2.1.6)$$

where $\theta, \beta, \sigma, \mu$ are complex parameters independent of n and ζ is given by the equations (2.1.3) and (2.1.4).

If we put $\theta = \{\alpha + (\beta + 1)mr\}$ and $\sigma = \{\lambda + \mu mr\}$ in Gould's identity (2.1.6), we get the first modified form of Gould's identity:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\}(\lambda + \mu n + \mu mr)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \\ &= (1 + \zeta)^{\{\alpha + (\beta + 1)mr\}} \left(\lambda + \mu mr + \frac{\mu\{\alpha + (\beta + 1)mr\}\zeta}{(1 - \beta\zeta)} \right) \end{aligned} \quad (2.1.7)$$

with

$$\zeta = t(1 + \zeta)^{\beta+1} \quad ; \zeta(0) = 0$$

If we put $\theta = \{\alpha + (\beta + 1)mr\}$ and $\sigma = \{\lambda + \mu(\beta + 1)r\}$ in Gould's identity (2.1.6), we get the second modified form of Gould's identity:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\}\{\lambda + \mu n + \mu(\beta + 1)r\}}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \\ &= (1 + \zeta)^{\{\alpha + (\beta + 1)mr\}} \left(\lambda + \mu(\beta + 1)r + \frac{\mu\{\alpha + (\beta + 1)mr\}\zeta}{(1 - \beta\zeta)} \right) \end{aligned} \quad (2.1.8)$$

with

$$\zeta = t(1 + \zeta)^{\beta+1} \quad ; \zeta(0) = 0$$

Gauss's Multiplication Theorem

For every positive integer m , we have

$$(b)_{mr} = m^{mr} \prod_{j=1}^m \binom{b+j-1}{m}_r \quad ; \quad r = 0, 1, 2, \dots \quad (2.1.9)$$

Summation identity[117, p.101, *Lemma*(3),(2.1.6)]

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} B(r, n) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} B(r, n + mr) \quad (2.1.10)$$

($\lfloor x \rfloor$ denotes the greatest integer in x ; $m \in \mathbb{N}$),

provided that series involved are absolutely convergent.

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ [85, p.200, 112(1)] are defined by

$$L_n^{(\alpha)}(x) = \frac{(1 + \alpha)_n}{n!} {}_1F_1 \left[-n; 1 + \alpha; x \right]; \quad n \in N_0 \quad (2.1.11)$$

Replacing α by $\alpha + n\beta$ in equation (2.1.11), we get

$${}_1F_1 \left[\begin{matrix} -n & ; \\ 1 + \alpha + n\beta & ; \end{matrix} x \right] = \frac{n!}{(1 + \alpha + n\beta)_n} L_n^{(\alpha + n\beta)}(x) \quad (2.1.12)$$

The Jacobi Polynomials of first kind $P_n^{(\alpha, \beta)}(x)$ [85, p.254 (132.1), p.255 (132.7)] are defined by the following equations:

$$P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1 + \alpha + \beta + n; & \frac{1 - x}{2} \\ 1 + \alpha & ; \end{matrix} \right] \quad (2.1.13)$$

$$P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha + \beta)_{2n}}{n!(1 + \alpha + \beta)_n} \left(\frac{x - 1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -\alpha - n; & \frac{2}{1 - x} \\ -\alpha - \beta - 2n; & \end{matrix} \right] \quad (2.1.14)$$

where n is a non-negative integer.

Replacing α by $(\alpha + bn)$ and β by $\{\beta - (b + 1)n\}$ in equation (2.1.13), we get

$${}_2F_1 \left[\begin{matrix} -n, 1 + \alpha + \beta; & \frac{1 - x}{2} \\ 1 + \alpha + bn & ; \end{matrix} \right] = \frac{n! \Gamma(\alpha + bn + 1)}{\Gamma\{\alpha + (b + 1)n + 1\}} P_n^{(\alpha + bn, \beta - (b + 1)n)}(x) \quad (2.1.15)$$

Replacing α by $(\alpha - n)$ and β by $\{\beta - (b + 1)n\}$ in equation (2.1.14), we get the following result

$${}_2F_1 \left[\begin{matrix} -n, -\alpha & ; & \frac{2}{1 - x} \\ bn - \alpha - \beta & ; & \end{matrix} \right] = \frac{\Gamma(1 + \alpha + \beta - bn - n)n!}{\Gamma(1 + \alpha + \beta - bn)} \left(\frac{2}{x - 1} \right)^n P_n^{(\alpha - n, \beta - bn - n)}(x) \quad (2.1.16)$$

The generalized Rice Polynomials $H_n^{(\alpha, \beta)}[\nu, \sigma, x]$ of Khandekar [67, p.158, eq.(2.3)] are defined by

$$H_n^{(\alpha, \beta)}[\nu, \sigma, x] = \binom{\alpha + n}{n} {}_3F_2 \left[\begin{matrix} -n, \alpha + \beta + n + 1, \nu; & x \\ \alpha + 1, \sigma & ; \end{matrix} \right] \quad (2.1.17)$$

$$H_n[\nu, \sigma, x] = H_n^{(0,0)}[\nu, \sigma, x] \quad (2.1.18)$$

$$P_n^{(\alpha, \beta)}(x) = H_n^{(\alpha, \beta)} \left[\nu, \nu, \frac{1-x}{2} \right] \quad (2.1.19)$$

Replacing α by $(\alpha + bn)$ and β by $\{\beta - (b+1)n\}$ in eq.(2.1.17), we get

$${}_3F_2 \left[\begin{matrix} -n, \nu, 1 + \alpha + \beta ; \\ 1 + \alpha + bn, \sigma ; \end{matrix} x \right] = \frac{n!}{(1 + \alpha + bn)_n} H_n^{(\alpha+bn, \beta-(b+1)n)} [\nu, \sigma, x] \quad (2.1.20)$$

Some useful Pochhammer's relations

$$\frac{(\lambda + \mu n) \left(\frac{\lambda + \mu n}{\mu(\beta+1-m)} + 1 \right)_r}{\left(\frac{\lambda + \mu n}{\mu(\beta+1-m)} \right)_r} = \lambda + \mu n + \mu(\beta+1)r - \mu m r \quad (2.1.21)$$

$$\left(\lambda + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \right) + \left(\mu m r + \frac{\mu \zeta \beta m r + \mu \zeta m r}{(1 - \beta \zeta)} \right) = \left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{(1 - \beta \zeta)} \right) \cdot \frac{\left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu m(\zeta+1)} + 1 \right)_r}{\left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu m(\zeta+1)} \right)_r} \quad (2.1.22)$$

$$\alpha + m(\beta+1)r = \alpha \frac{\left(\frac{\alpha}{m(\beta+1)} + 1 \right)_r}{\left(\frac{\alpha}{m(\beta+1)} \right)_r} \quad (2.1.23)$$

$$\begin{aligned} \left(\lambda + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \right) + \left\{ \mu(\beta+1)r + \frac{\mu \zeta \beta m r}{(1 - \beta \zeta)} + \frac{\mu \zeta m r}{(1 - \beta \zeta)} \right\} \\ = \frac{\{\lambda(1 - \beta \zeta) + \mu \zeta \alpha\}}{(1 - \beta \zeta)} \frac{\left(\frac{\{\lambda(1 - \beta \zeta) + \mu \zeta \alpha\}}{\mu(\beta+1)(1 - \beta \zeta + m \zeta)} + 1 \right)_r}{\left(\frac{\{\lambda(1 - \beta \zeta) + \mu \zeta \alpha\}}{\mu(\beta+1)(1 - \beta \zeta + m \zeta)} \right)_r} \end{aligned} \quad (2.1.23a)$$

where

$$r = 0, 1, 2, 3, \dots$$

Now we shall discuss some special cases of the implicit functions defined by equation (2.1.3) subject to the condition (2.1.4). Using Mathematica 9.0, we can find the roots of resulting cubic equation in ζ for different values of β in equation (2.1.3).

Case I:- When $\beta = 0$ in (2.1.3), then particular value of ζ (satisfying the condition (2.1.4)) is denoted by

$$\Theta = \frac{t}{1-t} \quad (2.1.24)$$

Case II:- When $\beta = 1$ in (2.1.3), we get

$$t\zeta^2 + (2t - 1)\zeta + t = 0$$

then one of the values of ζ (satisfying the condition (2.1.4)) is given by

$$\Lambda = \frac{1 - 2t - \sqrt{(1 - 4t)}}{2t} \quad (2.1.25)$$

Case III:- When $\beta = -2$ in (2.1.3), we get

$$\zeta^2 + \zeta - t = 0$$

then the particular value of ζ (satisfying the condition (2.1.4)) is given by

$$\Xi = \frac{-1 + \sqrt{(1 + 4t)}}{2} \quad (2.1.26)$$

Case IV:- When $\beta = -3$ in (2.1.3), we get

$$\zeta^3 + 2\zeta^2 + \zeta - t = 0$$

then one of the roots (satisfying the condition (2.1.4)) of above equation is given by

$$\Upsilon = \frac{1}{3} \left[-2 + \frac{2^{\frac{1}{3}}}{\left\{ 2 + 27t + 3\sqrt{3}\sqrt{(4t + 27t^2)} \right\}^{\frac{1}{3}}} + \frac{\left\{ 2 + 27t + 3\sqrt{3}\sqrt{(4t + 27t^2)} \right\}^{\frac{1}{3}}}{2^{\frac{1}{3}}} \right] \quad (2.1.27)$$

Case V:- When $\beta = -\frac{1}{2}$ in (2.1.3), we get

$$\zeta^2 - t^2\zeta - t^2 = 0$$

then one of the roots (satisfying the condition (2.1.4)) of above equation is given by

$$U = \frac{t}{2} \{ t + \sqrt{(t^2 + 4)} \} \quad (2.1.28)$$

Case VI:- When $\beta = \frac{-3}{2}$ in (2.1.3), we get

$$\zeta^3 + \zeta^2 - t^2 = 0$$

then one of the roots (satisfying the condition (2.1.4)) of above equation is given by

$$\Psi = -\frac{1}{3} \left[1 + \frac{(1 + \iota\sqrt{3})}{2^{\frac{2}{3}} \left\{ -2 + 27t^2 + 3\sqrt{3}\sqrt{(-4t^2 + 27t^4)} \right\}^{\frac{1}{3}}} + \frac{(1 - \iota\sqrt{3}) \left\{ -2 + 27t^2 + 3\sqrt{3}\sqrt{(-4t^2 + 27t^4)} \right\}^{\frac{1}{3}}}{2^{\frac{4}{3}}} \right] \quad (2.1.29)$$

where $\iota = \sqrt{(-1)}$.

Case VII:- When $\beta = -\frac{1}{3}$ in (2.1.3), we obtain

$$\zeta^3 - t^3\zeta^2 - 2t^3\zeta - t^3 = 0$$

then one of the values of ζ (satisfying the condition (2.1.4)) is denoted by

$$\Pi = \frac{t^3}{3} - \frac{2^{\frac{1}{3}}(-6t^3 - t^6)}{3\{27t^3 + 18t^6 + 2t^9 + 3\sqrt{3}\sqrt{(27t^6 + 4t^9)}\}^{\frac{1}{3}}} + \frac{\{27t^3 + 18t^6 + 2t^9 + 3\sqrt{3}\sqrt{(27t^6 + 4t^9)}\}^{\frac{1}{3}}}{3 \cdot 2^{\frac{1}{3}}} \quad (2.1.30)$$

Case VIII:- When $\beta = -\frac{2}{3}$ in (2.1.3), we obtain

$$\zeta^3 - t^3\zeta - t^3 = 0$$

then one of the values of ζ (satisfying the condition (2.1.4)) is denoted by

$$\Phi = \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}} t^3}{\left\{9t^3 + \sqrt{3}\sqrt{(27t^6 - 4t^9)}\right\}^{\frac{1}{3}}} + \frac{\left\{9t^3 + \sqrt{3}\sqrt{(27t^6 - 4t^9)}\right\}^{\frac{1}{3}}}{2^{\frac{1}{3}} 3^{\frac{2}{3}}} \quad (2.1.31)$$

2.2 Main Generating Relations

First Generating Relation:

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n = (1 + \zeta)^\alpha.$$

$$\cdot \left\{ \frac{\lambda(1-\beta\zeta) + \mu\zeta\alpha}{\alpha(1-\beta\zeta)} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta) + \mu\zeta\alpha}{m\mu(1+\zeta)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta) + \mu\zeta\alpha}{m\mu(1+\zeta)}, (b_q); \end{matrix} \middle| x(-\zeta)^m \right] \quad (2.2.1)$$

where

$$\zeta = t(1 + \zeta)^{1+\beta}; \zeta(0) = 0.$$

provided that involved series on both sides are absolutely convergent.

Here Srivastava's generalized hypergeometric polynomials of one variable $H_n^{(\alpha, \beta)}(x; m)$ [117, p.360, eq.(7.3.3); see also 109, pp.331 – 332] are given by

$$H_n^{(\alpha, \beta)}(x; m) = \binom{\alpha + (\beta + 1)n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) \\ \Delta(m; 1 + \alpha + \beta n), (b_q); \end{matrix} \middle| x \right] \quad (2.2.2)$$

where α, β are complex parameters independent of ' n ' and $\Delta(m; \lambda)$ abbreviates the array of m number of parameters given by

$$\frac{\lambda}{m}, \frac{\lambda + 1}{m}, \dots, \frac{\lambda + m - 1}{m}; m \in \mathbb{N}$$

Independent Demonstration:

Using the definition (2.2.2) of $H_n^{(\alpha, \beta)}(x; m)$ and then the power series form of ${}_{p+m}F_{q+m}[x]$ in left hand side of equation (2.2.1), we get

$$\Omega = \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n = \sum_{n=0}^{\infty} \frac{(\lambda + \mu n) \Gamma\{\alpha + (\beta + 1)n + 1\}}{\{\alpha + (\beta + 1)n\} \Gamma(n + 1) \Gamma\{\alpha + \beta n + 1\}} \cdot \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{\prod_{j=1}^m \left(\frac{-n + j - 1}{m} \right)_r [(a_p)]_r x^r t^n}{\prod_{j=1}^m \left(\frac{1 + \alpha + n\beta + j - 1}{m} \right)_r [(b_q)]_r r!}$$

Using Gauss's multiplication theorem (2.1.9) in above equation, we get

$$\Omega = \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(\lambda + \mu n) (-n)_{mr} (\alpha)_{n(\beta+1)} [(a_p)]_r x^r t^n}{\alpha (\alpha + 1)_{n\beta} (1 + \alpha + n\beta)_{mr} [(b_q)]_r r! n!} \quad (2.2.3)$$

Now applying summation identity (2.1.10) and then simplifying further, we get

$$\Omega = \sum_{r=0}^{\infty} \frac{(\alpha)_{m(\beta+1)r} [(a_p)]_r x^r (-t)^{mr}}{\alpha (\alpha + 1)_{m(\beta+1)r} [(b_q)]_r r!} \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} (\lambda + \mu n + \mu mr)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \cdot \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \quad (2.2.4)$$

Now using first modified Gould's identity (2.1.7), we get

$$\Omega = (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \frac{\left(\frac{\alpha}{m(\beta+1)}\right)_r [(a_p)]_r x^r (-t)^{mr}}{\alpha \left(\frac{\alpha}{m(\beta+1)} + 1\right)_r [(b_q)]_r r!} (1 + \zeta)^{m(\beta+1)r} \cdot \left[\left(\lambda + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \right) + \left(\mu m r + \frac{\mu \zeta (\beta + 1) m r}{(1 - \beta \zeta)} \right) \right] \quad (2.2.5)$$

Now using (2.1.22) and simplifying it further, we get

$$\Omega = (1 + \zeta)^\alpha \left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{(1 - \beta \zeta)} \right) \sum_{r=0}^{\infty} \frac{\left(\frac{\alpha}{m(\beta+1)}\right)_r [(a_p)]_r x^r (-\zeta)^{mr}}{\alpha \left(\frac{\alpha}{m(\beta+1)} + 1\right)_r [(b_q)]_r r!} \cdot \frac{\left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu m(1 + \zeta)} + 1\right)_r}{\left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu m(1 + \zeta)}\right)_r} \quad (2.2.6)$$

After solving it further, we get the result (2.2.1) in the form of a generalized hypergeometric function of one variable.

Second Generating Relation:

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n = (1 + \zeta)^\alpha \cdot \left\{ \frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\alpha(1 - \beta \zeta)} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu(1 + \beta)(1 - \beta \zeta + m \zeta)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu(1 + \beta)(1 - \beta \zeta + m \zeta)}, (b_q); \end{matrix} x(-\zeta)^m \right] \quad (2.2.7)$$

where

$$\zeta = t(1 + \zeta)^{1+\beta}; \zeta(0) = 0.$$

provided that involved series on both sides are absolutely convergent.

Here we define new generalized hypergeometric polynomials $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$, known as "Pathan's generalized hypergeometric polynomials of one variable", given by

$$\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) = \binom{\alpha + (\beta + 1)n}{n}_{p+m+1} F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(\beta+1-m)}, (a_p); \\ \Delta(m; 1 + \alpha + \beta n), \frac{\lambda + \mu n}{\mu(\beta+1-m)}, (b_q); \end{matrix} x \right] \quad (2.2.8)$$

where α, β are complex parameters independent of 'n' and $\Delta(m; \lambda)$ abbreviates the array of m number of parameters given by

$$\frac{\lambda}{m}, \frac{\lambda + 1}{m}, \dots, \frac{\lambda + m - 1}{m}; m \in \mathbb{N}$$

Independent Demonstration:

Using the definition (2.2.8) of $\mathcal{B}_n^{(\alpha,\beta)}(x; m, \lambda, \mu)$ and then the power series form of ${}_{p+m+1}F_{q+m+1}[x]$ in left hand side of equation (2.2.7), using Gauss's multiplication theorem (2.1.9) and result (2.1.21), we get

$$\begin{aligned} \Omega^* &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha,\beta)}(x; m, \lambda, \mu) t^n = \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{[\lambda + \mu n + \mu(\beta + 1)r - \mu mr] (-n)_{mr} \Gamma\{\alpha + n(\beta + 1)\} [(a_p)]_r x^r t^n}{\Gamma\{\alpha + n\beta + 1 + mr\} [(b_q)]_r r! n!} \end{aligned} \quad (2.2.9)$$

Now applying summation identity (2.1.10) in above equation then simplifying further, we get

$$\begin{aligned} \Omega^* &= \sum_{r=0}^{\infty} \frac{[(a_p)]_r x^r (-t)^{mr}}{[\alpha + m(\beta + 1)r] [(b_q)]_r r!} \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} (\lambda + \mu n + \mu(\beta + 1)r)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \\ &\quad \cdot \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \end{aligned} \quad (2.2.10)$$

Now using second modified Gould's identity (2.1.8) and eq.(2.1.23), we get

$$\begin{aligned} \Omega^* &= (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \frac{\left(\frac{\alpha}{m(\beta+1)}\right)_r [(a_p)]_r x^r (-\zeta)^{mr}}{\alpha \left(\frac{\alpha}{m(\beta+1)} + 1\right)_r [(b_q)]_r r!} \\ &\quad \cdot \left[\left(\lambda + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \right) + \left\{ \mu(\beta + 1)r + \frac{\mu \zeta (\beta + 1) mr}{(1 - \beta \zeta)} \right\} \right] \end{aligned} \quad (2.2.11)$$

Now using equation (2.1.23a) in above equation and summing it up into hypergeometric form further, we get the desired result (2.2.7).

2.3 Known Applications of First Generating Relation (2.2.1)

(i). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ in equation (2.2.1) and after simplifying, we get

$$\sum_{n=0}^{\infty} H_n^{(\alpha,\beta)}(x; m) t^n = \frac{(1 + \zeta)^{\alpha+1}}{(1 - \beta \zeta)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta)^m \right] \quad (2.3.1)$$

which is the result of Srivastava [106, p.975; see also 107, p.233, eq.(12)]. Here ζ being given by equation (2.1.3) and (2.1.4) and $H_n^{(\alpha, \beta)}(x; m)$ is given by equation (2.2.2).

(ii). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ and $m = 1$ in equation (2.2.1) and using the definition of $H_n^{(\alpha, \beta)}(x; 1)$ and after simplification, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n}_{p+1} F_{q+1} \left[\begin{matrix} -n, (a_p) \\ 1 + \alpha + n\beta, (b_q) \end{matrix} ; x \right] t^n \\ = \frac{(1+\zeta)^{\alpha+1}}{(1-\beta\zeta)^p} {}_pF_q \left[\begin{matrix} (a_p) \\ (b_q) \end{matrix} ; x(-\zeta) \right] \end{aligned} \quad (2.3.2)$$

which is the result of Srivastava [105, p.591, eq.(9); see also 59, p.1186]. ζ is given by equations (2.1.3) and (2.1.4).

(iii). Putting $\lambda = 1, \mu = \frac{1}{2\alpha}, \beta = \frac{-1}{2}$ and $m = 1$ in equation (2.2.1), using the definition (2.2.2) of $H_n^{(\alpha, -\frac{1}{2})}(x; 1) = f_n^{(\alpha)}(x)$ and replacing ζ by U , we get

$$\begin{aligned} \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n = \sum_{n=0}^{\infty} \binom{\alpha + \frac{n}{2}}{n}_{p+1} F_{q+1} \left[\begin{matrix} -n, (a_p) \\ 1 + \alpha - \frac{n}{2}, (b_q) \end{matrix} ; x \right] t^n \\ = \frac{[1+U]^{\alpha+1}}{(1+\frac{U}{2})^p} {}_pF_q \left[\begin{matrix} (a_p) \\ (b_q) \end{matrix} ; -xU \right] \end{aligned} \quad (2.3.3)$$

where

$$U = \frac{t}{2} \{t + \sqrt{t^2 + 4}\}$$

which is the known result of Brown [7, p.264, eq.(7)] and $f_n^{(\alpha)}(x)$ is Brown's generalized hypergeometric polynomial [117, p.358, eq.(7.2.4)].

2.4 New Applications of First Generating Relation (2.2.1)

(i). Putting $\beta = 0$ and $\zeta = \Theta = \frac{t}{1-t}$ from equation (2.1.24) in equation (2.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + n)} \binom{\alpha + n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) \\ \Delta(m; 1 + \alpha), (b_q) \end{matrix} ; x \right] t^n = (1-t)^{-\alpha-1}.$$

$$\cdot \left\{ \frac{\lambda(1-t) + \mu\alpha t}{\alpha} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{\alpha}{m}, 1 + \frac{\lambda(1-t) + \mu\alpha t}{m\mu}, (a_p); \\ 1 + \frac{\alpha}{m}, \frac{\lambda(1-t) + \mu\alpha t}{m\mu}, (b_q); \end{matrix} x \left(\frac{-t}{1-t} \right)^m \right] \quad (2.4.1)$$

(ii). Putting $\beta = 1$ and $\zeta = \Lambda = \frac{1-2t-\sqrt{(1-4t)}}{2t}$ from equation (2.1.25) in equation (2.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + 2n)} \binom{\alpha + 2n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; \\ \Delta(m; 1 + \alpha + n), (b_q); \end{matrix} x \right] t^n = (1 + \Lambda)^\alpha.$$

$$\cdot \left\{ \frac{\lambda(1-\Lambda) + \mu\alpha\Lambda}{\alpha(1-\Lambda)} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{\alpha}{2m}, 1 + \frac{\lambda(1-\Lambda) + \mu\alpha\Lambda}{m\mu(1+\Lambda)}, (a_p); \\ 1 + \frac{\alpha}{2m}, \frac{\lambda(1-\Lambda) + \mu\alpha\Lambda}{m\mu(1+\Lambda)}, (b_q); \end{matrix} x(-\Lambda)^m \right] \quad (2.4.2)$$

(iii). Putting $\beta = -2$ and $\zeta = \Xi = \frac{-1+\sqrt{(1+4t)}}{2}$ from equation (2.1.26) in equation (2.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - n)} \binom{\alpha - n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; \\ \Delta(m; 1 + \alpha - 2n), (b_q); \end{matrix} x \right] t^n = (1 + \Xi)^\alpha.$$

$$\cdot \left\{ \frac{\lambda(1+2\Xi) + \mu\alpha\Xi}{\alpha(1+2\Xi)} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{-\alpha}{m}, 1 + \frac{\lambda(1+2\Xi) + \mu\alpha\Xi}{m\mu(1+\Xi)}, (a_p); \\ 1 - \frac{\alpha}{m}, \frac{\lambda(1+2\Xi) + \mu\alpha\Xi}{m\mu(1+\Xi)}, (b_q); \end{matrix} x(-\Xi)^m \right] \quad (2.4.3)$$

(iv). Putting $\beta = -3$ and $\zeta = \Upsilon$ from equation (2.1.27) in equation (2.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - 2n)} \binom{\alpha - 2n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; \\ \Delta(m; 1 + \alpha - 3n), (b_q); \end{matrix} x \right] t^n = (1 + \Upsilon)^\alpha.$$

$$\cdot \left\{ \frac{\lambda(1+3\Upsilon) + \mu\alpha\Upsilon}{\alpha(1+3\Upsilon)} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{-\alpha}{2m}, 1 + \frac{\lambda(1+3\Upsilon) + \mu\alpha\Upsilon}{m\mu(1+\Upsilon)}, (a_p); \\ 1 - \frac{\alpha}{2m}, \frac{\lambda(1+3\Upsilon) + \mu\alpha\Upsilon}{m\mu(1+\Upsilon)}, (b_q); \end{matrix} x(-\Upsilon)^m \right] \quad (2.4.4)$$

(v). Putting $\beta = \frac{-1}{2}$ and $\zeta = U$ from equation (2.1.28) in equation (2.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{2}n)} \binom{\alpha + \frac{1}{2}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; \\ \Delta(m; 1 + \alpha - \frac{1}{2}n), (b_q); \end{matrix} x \right] t^n = (1 + U)^\alpha.$$

$$\cdot \left\{ \frac{\lambda(1 + \frac{U}{2}) + \alpha\mu U}{\alpha(1 + \frac{U}{2})} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{2\alpha}{m}, 1 + \frac{\lambda(1 + \frac{U}{2}) + \alpha\mu U}{m\mu(1+U)}, (a_p); \\ 1 + \frac{2\alpha}{m}, \frac{\lambda(1 + \frac{U}{2}) + \alpha\mu U}{m\mu(1+U)}, (b_q); \end{matrix} x(-U)^m \right] \quad (2.4.5)$$

(vi). Putting $\beta = \frac{-3}{2}$ and $\zeta = \Psi$ from equation (2.1.29) in equation (2.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - \frac{1}{2}n)} \binom{\alpha - \frac{1}{2}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - \frac{3}{2}n), (b_q); \end{matrix} \right] t^n = (1 + \Psi)^\alpha.$$

$$\cdot \left\{ \frac{\lambda(1 + \frac{3}{2}\Psi) + \alpha\mu\Psi}{\alpha(1 + \frac{3}{2}\Psi)} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{-2\alpha}{m}, 1 + \frac{\lambda(1 + \frac{3}{2}\Psi) + \alpha\mu\Psi}{m\mu(1 + \Psi)}, (a_p); & x(-\Psi)^m \\ 1 - \frac{2\alpha}{m}, \frac{\lambda(1 + \frac{3}{2}\Psi) + \alpha\mu\Psi}{m\mu(1 + \Psi)}, (b_q); \end{matrix} \right] \quad (2.4.6)$$

(vii). Putting $\beta = \frac{-1}{3}$ and $\zeta = \Pi$ from equation (2.1.30) in equation (2.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{2}{3}n)} \binom{\alpha + \frac{2}{3}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - \frac{1}{3}n), (b_q); \end{matrix} \right] t^n = (1 + \Pi)^\alpha.$$

$$\cdot \left\{ \frac{\lambda(1 + \frac{1}{3}\Pi) + \alpha\mu\Pi}{\alpha(1 + \frac{1}{3}\Pi)} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{3\alpha}{2m}, 1 + \frac{\lambda(1 + \frac{1}{3}\Pi) + \alpha\mu\Pi}{m\mu(1 + \Pi)}, (a_p); & x(-\Pi)^m \\ 1 + \frac{3\alpha}{2m}, \frac{\lambda(1 + \frac{1}{3}\Pi) + \alpha\mu\Pi}{m\mu(1 + \Pi)}, (b_q); \end{matrix} \right] \quad (2.4.7)$$

(viii). Putting $\beta = \frac{-2}{3}$ and $\zeta = \Phi$ from equation (2.1.31) in equation (2.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{3}n)} \binom{\alpha + \frac{1}{3}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - \frac{2}{3}n), (b_q); \end{matrix} \right] t^n = (1 + \Phi)^\alpha.$$

$$\cdot \left\{ \frac{\lambda(1 + \frac{2}{3}\Phi) + \mu\alpha\Phi}{\alpha(1 + \frac{2}{3}\Phi)} \right\}_{p+2} F_{q+2} \left[\begin{matrix} \frac{3\alpha}{m}, 1 + \frac{\lambda(1 + \frac{2}{3}\Phi) + \mu\alpha\Phi}{m\mu(1 + \Phi)}, (a_p); & x(-\Phi)^m \\ 1 + \frac{3\alpha}{m}, \frac{\lambda(1 + \frac{2}{3}\Phi) + \mu\alpha\Phi}{m\mu(1 + \Phi)}, (b_q); \end{matrix} \right] \quad (2.4.8)$$

2.5 New Applications of Second Generating Relation (2.2.7)

(i). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ in equation (2.2.7) and after simplifying, we get

$$\sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n}_{p+m+1} F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\alpha + (\beta+1)n}{(\beta+1)(\beta+1-m)}, (a_p); & x \\ \Delta(m; 1 + \alpha + \beta n), \frac{\alpha + (\beta+1)n}{(\beta+1)(\beta+1-m)}, (b_q); \end{matrix} \right] t^n$$

$$= \frac{(1 + \zeta)^{(\alpha+1)}}{(1 - \beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\alpha(1+\zeta)}{(\beta+1)^2(1-\beta\zeta+m\zeta)}, (a_p); & x(-\zeta)^m \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\alpha(1+\zeta)}{(\beta+1)^2(1-\beta\zeta+m\zeta)}, (b_q); \end{matrix} \right] \quad (2.5.1)$$

where ζ is given by equations (2.1.3) and (2.1.4).

(ii). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}, m = 1$ in equation (2.2.7) and using the definition of $\mathcal{B}_n^{(\alpha, \beta)}(x; 1, 1, \frac{\beta+1}{\alpha})$, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n}_{p+2} F_{q+2} \left[\begin{matrix} -n, 1 + \frac{\alpha + (\beta+1)n}{(\beta+1)\beta}, (a_p); \\ 1 + \alpha + \beta n, \frac{\alpha + (\beta+1)n}{(\beta+1)\beta}, (b_q); \end{matrix} x \right] t^n \\ &= \frac{(1+\zeta)^{(\alpha+1)}}{(1-\beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{(\beta+1)}, 1 + \frac{\alpha(1+\zeta)}{(\beta+1)^2(1-\beta\zeta+\zeta)}, (a_p); \\ 1 + \frac{\alpha}{(\beta+1)}, \frac{\alpha(1+\zeta)}{(\beta+1)^2(1-\beta\zeta+\zeta)}, (b_q); \end{matrix} x(-\zeta) \right] \end{aligned} \quad (2.5.2)$$

where ζ is given by equations (2.1.3) and (2.1.4).

(iii). Putting $\lambda = 1, \mu = \frac{1}{2\alpha}, \beta = \frac{-1}{2}, m = 1$ in equation (2.2.7), using the definition of $\mathcal{B}_n^{(\alpha, -\frac{1}{2})}(x; 1, 1, \frac{1}{2\alpha})$ and replacing ζ by U from equation (2.1.28), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\alpha + \frac{n}{2}}{n}_{p+2} F_{q+2} \left[\begin{matrix} -n, 1 - 4\alpha - 2n, (a_p); \\ 1 + \alpha - \frac{n}{2}, -4\alpha - 2n, (b_q); \end{matrix} x \right] t^n \\ &= \frac{(1+U)^{\alpha+1}}{(1+\frac{U}{2})} {}_{p+2}F_{q+2} \left[\begin{matrix} 2\alpha, 1 + \frac{8\alpha(1+U)}{(2+3U)}, (a_p); \\ 1 + 2\alpha, \frac{8\alpha(1+U)}{(2+3U)}, (b_q); \end{matrix} x(-U) \right] \end{aligned} \quad (2.5.3)$$

(iv). Putting $\beta = 0$ and $\zeta = \Theta = \frac{t}{1-t}$ from equation (2.1.24) in equation (2.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + n)} \binom{\alpha + n}{n}_{p+m+1} F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(1-m)}, (a_p); \\ \Delta(m; 1 + \alpha), \frac{\lambda + \mu n}{\mu(1-m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1-t)^{-\alpha-1} \left\{ \frac{\lambda(1-t) + \mu\alpha t}{\alpha} \right\} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m}, 1 + \frac{\lambda(1-t) + \mu\alpha t}{\mu(1-t+mt)}, (a_p); \\ 1 + \frac{\alpha}{m}, \frac{\lambda(1-t) + \mu\alpha t}{\mu(1-t+mt)}, (b_q); \end{matrix} x \left(\frac{t}{t-1} \right)^m \right] \end{aligned} \quad (2.5.4)$$

(v). Putting $\beta = 1$ and $\zeta = \Lambda = \frac{1-2t-\sqrt{(1-4t)}}{2t}$ from equation (2.1.25) in equation (2.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + 2n)} \binom{\alpha + 2n}{n}_{p+m+1} F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(2-m)}, (a_p); \\ \Delta(m; 1 + \alpha + n), \frac{\lambda + \mu n}{\mu(2-m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1+\Lambda)^{\alpha} \left\{ \frac{\lambda(1-\Lambda) + \mu\alpha\Lambda}{\alpha(1-\Lambda)} \right\} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{2m}, 1 + \frac{\lambda(1-\Lambda) + \mu\alpha\Lambda}{2\mu(1-\Lambda+m\Lambda)}, (a_p); \\ 1 + \frac{\alpha}{2m}, \frac{\lambda(1-\Lambda) + \mu\alpha\Lambda}{2\mu(1-\Lambda+m\Lambda)}, (b_q); \end{matrix} x(-\Lambda)^m \right] \end{aligned} \quad (2.5.5)$$

(vi). Putting $\beta = -2$ and $\zeta = \Xi = \frac{-1+\sqrt{(1+4t)}}{2}$ from equation (2.1.26) in equation (2.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - n)} \binom{\alpha - n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 - \frac{(\lambda + \mu n)}{\mu(1+m)}, (a_p); \\ \Delta(m; 1 + \alpha - 2n), -\frac{(\lambda + \mu n)}{\mu(1+m)}, (b_q); \end{array} x \right] t^n \\ &= (1 + \Xi)^\alpha \left\{ \frac{\lambda(1 + 2\Xi) + \mu\alpha\Xi}{\alpha(1 + 2\Xi)} \right\}_{p+2} F_{q+2} \left[\begin{array}{c} \frac{-\alpha}{m}, 1 - \frac{\lambda(1+2\Xi) + \mu\alpha\Xi}{\mu(1+2\Xi+m\Xi)}, (a_p); \\ 1 - \frac{\alpha}{m}, -\frac{\{\lambda(1+2\Xi) + \mu\alpha\Xi\}}{\mu(1+2\Xi+m\Xi)}, (b_q); \end{array} x(-\Xi)^m \right] \end{aligned} \quad (2.5.6)$$

(vii). Putting $\beta = -3$ and $\zeta = \Upsilon$ from equation (2.1.27) in equation (2.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - 2n)} \binom{\alpha - 2n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 - \frac{(\lambda + \mu n)}{\mu(2+m)}, (a_p); \\ \Delta(m; 1 + \alpha - 3n), -\frac{(\lambda + \mu n)}{\mu(2+m)}, (b_q); \end{array} x \right] t^n \\ &= (1 + \Upsilon)^\alpha \left\{ \frac{\lambda(1 + 3\Upsilon) + \mu\alpha\Upsilon}{\alpha(1 + 3\Upsilon)} \right\}_{p+2} F_{q+2} \left[\begin{array}{c} \frac{-\alpha}{2m}, 1 - \frac{\lambda(1+3\Upsilon) + \mu\alpha\Upsilon}{2\mu(1+3\Upsilon+m\Upsilon)}, (a_p); \\ 1 - \frac{\alpha}{2m}, -\frac{\{\lambda(1+3\Upsilon) + \mu\alpha\Upsilon\}}{2\mu(1+3\Upsilon+m\Upsilon)}, (b_q); \end{array} x(-\Upsilon)^m \right] \end{aligned} \quad (2.5.7)$$

(viii). Putting $\beta = \frac{-1}{2}$ and $\zeta = U$ from equation (2.1.28) in equation (2.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{2}n)} \binom{\alpha + \frac{1}{2}n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 + \frac{(\lambda + \mu n)}{\mu(\frac{1}{2}-m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{1}{2}n), \frac{(\lambda + \mu n)}{\mu(\frac{1}{2}-m)}, (b_q); \end{array} x \right] t^n \\ &= (1 + U)^\alpha \left\{ \frac{\lambda(1 + \frac{U}{2}) + \alpha\mu U}{\alpha(1 + \frac{U}{2})} \right\}_{p+2} F_{q+2} \left[\begin{array}{c} \frac{2\alpha}{m}, 1 + \frac{2\lambda(1+\frac{U}{2}) + \alpha\mu U}{\mu(1+\frac{U}{2}+mU)}, (a_p); \\ 1 + \frac{2\alpha}{m}, \frac{2\lambda(1+\frac{U}{2}) + \alpha\mu U}{\mu(1+\frac{U}{2}+mU)}, (b_q); \end{array} x(-U)^m \right] \end{aligned} \quad (2.5.8)$$

(ix). Putting $\beta = \frac{-3}{2}$ and $\zeta = \Psi$ from equation (2.1.29) in equation (2.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - \frac{1}{2}n)} \binom{\alpha - \frac{1}{2}n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 - \frac{(\lambda + \mu n)}{\mu(\frac{1}{2}+m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{3}{2}n), -\frac{(\lambda + \mu n)}{\mu(\frac{1}{2}+m)}, (b_q); \end{array} x \right] t^n \\ &= (1 + \Psi)^\alpha \left\{ \frac{\lambda(1 + \frac{3}{2}\Psi) + \alpha\mu\Psi}{\alpha(1 + \frac{3}{2}\Psi)} \right\}_{p+2} F_{q+2} \left[\begin{array}{c} \frac{-2\alpha}{m}, 1 - 2\frac{\lambda(1+\frac{3}{2}\Psi) + \alpha\mu\Psi}{\mu(1+\frac{3}{2}\Psi+m\Psi)}, (a_p); \\ 1 - \frac{2\alpha}{m}, -\frac{\{2\lambda(1+\frac{3}{2}\Psi) + \alpha\mu\Psi\}}{\mu(1+\frac{3}{2}\Psi+m\Psi)}, (b_q); \end{array} x(-\Psi)^m \right] \end{aligned} \quad (2.5.9)$$

(x). Putting $\beta = \frac{-1}{3}$ and $\zeta = \Pi$ from equation (2.1.30) in equation (2.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{2}{3}n)} \binom{\alpha + \frac{2}{3}n}{n} {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{(\lambda + \mu n)}{\mu(\frac{2}{3}-m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{1}{3}n), \frac{(\lambda + \mu n)}{\mu(\frac{2}{3}-m)}, (b_q); \end{matrix} \right] x t^n \\ &= (1+\Pi)^\alpha \left\{ \frac{\lambda(1 + \frac{1}{3}\Pi) + \alpha\mu\Pi}{\alpha(1 + \frac{1}{3}\Pi)} \right\} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{3\alpha}{2m}, 1 + \frac{3\{\lambda(1 + \frac{1}{3}\Pi) + \alpha\mu\Pi\}}{2\mu(1 + \frac{1}{3}\Pi + m\Pi)}, (a_p); \\ 1 + \frac{3\alpha}{2m}, \frac{3\{\lambda(1 + \frac{1}{3}\Pi) + \alpha\mu\Pi\}}{2\mu(1 + \frac{1}{3}\Pi + m\Pi)}, (b_q); \end{matrix} \right] x(-\Pi)^m \end{aligned} \quad (2.5.10)$$

(xi). Putting $\beta = \frac{-2}{3}$ and $\zeta = \Phi$ from equation (2.1.31) in equation (2.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{3}n)} \binom{\alpha + \frac{1}{3}n}{n} {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{(\lambda + \mu n)}{\mu(\frac{1}{3}-m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{2}{3}n), \frac{(\lambda + \mu n)}{\mu(\frac{1}{3}-m)}, (b_q); \end{matrix} \right] x t^n \\ &= (1+\Phi)^\alpha \left\{ \frac{\lambda(1 + \frac{2}{3}\Phi) + \mu\alpha\Phi}{\alpha(1 + \frac{2}{3}\Phi)} \right\} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{3\alpha}{m}, 1 + \frac{3\{\lambda(1 + \frac{2}{3}\Phi) + \mu\alpha\Phi\}}{\mu(1 + \frac{2}{3}\Phi + m\Phi)}, (a_p); \\ 1 + \frac{3\alpha}{m}, \frac{3\{\lambda(1 + \frac{2}{3}\Phi) + \mu\alpha\Phi\}}{\mu(1 + \frac{2}{3}\Phi + m\Phi)}, (b_q); \end{matrix} \right] x(-\Phi)^m \end{aligned} \quad (2.5.11)$$

Making suitable adjustments of parameters and variables in all generating relations of sections 2.4 and 2.5, we can also obtain a number of new generating relations involving restricted generalized Laguerre polynomials, restricted Jacobi polynomials, restricted generalized Rice polynomials of Khandekar and other orthogonal polynomials.

Chapter 3

Certain Hypergeometric Generating Relations via Gould's Identity (Part-II)

3.1 Introduction

The present chapter reveals some more hypergeometric generating relations associated with two hypergeometric polynomials of one variable $H_n^{(\alpha, \beta)}(x; m)$ and $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ with their independent demonstrations via Gould's identity. Some well known and new generating relations are deduced in the form of particular cases.

Some Useful Pochhammer's Relations

$$\frac{\lambda + \mu(\beta + 1)r}{\alpha + m(\beta + 1)r} = \frac{\lambda \left(\frac{\lambda}{\mu(\beta + 1)} + 1 \right)_r \left(\frac{\alpha}{m(\beta + 1)} \right)_r}{\alpha \left(\frac{\lambda}{\mu(\beta + 1)} \right)_r \left(\frac{\alpha}{m(\beta + 1)} + 1 \right)_r} \quad (3.1.1)$$

$$\frac{\{\mu \zeta \alpha + \mu \zeta \beta m r + \mu \zeta m r\}}{(1 - \beta \zeta)} = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\alpha}{m(\beta + 1)} + 1 \right)_r}{\left(\frac{\alpha}{m(\beta + 1)} \right)_r} \quad (3.1.2)$$

where

$$r = 0, 1, 2, 3, \dots$$

3.2 Main Generating Relations

First Generating Relation:

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n = (1 + \zeta)^\alpha.$$

$$\left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta + 1)}, 1 + \frac{\lambda}{m\mu}, (a_p); \\ 1 + \frac{\alpha}{m(\beta + 1)}, \frac{\lambda}{m\mu}, (b_q); \end{matrix} \right. x(-\zeta)^m \right] + \frac{\mu \zeta}{(1 - \beta \zeta)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (3.2.1)$$

where

$$\zeta = t(1 + \zeta)^{1+\beta}; \zeta(0) = 0.$$

provided that involved series on both sides are absolutely convergent.

Here Srivastava's generalized hypergeometric polynomials $H_n^{(\alpha, \beta)}(x; m)$

[117, p.360, eq.(7.3.3); see also 109, pp.331 – 332] are given by equation (2.2.2).

Independent Demonstration:

Using the definition (2.2.2) of $H_n^{(\alpha,\beta)}(x;m)$ and then the power series form of ${}_{p+m}F_{q+m}[x]$ in left hand side of equation (3.2.1), we get

$$\Omega = \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha,\beta)}(x;m) t^n = \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \cdot \frac{\Gamma\{\alpha + (\beta + 1)n + 1\}}{\Gamma(n+1) \Gamma\{\alpha + \beta n + 1\}} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{\prod_{j=1}^m \left(\frac{-n+j-1}{m} \right)_r [(a_p)]_r x^r t^n}{\prod_{j=1}^m \left(\frac{1+\alpha+n\beta+j-1}{m} \right)_r [(b_q)]_r r!} \quad (3.2.2)$$

Using Gauss's multiplication theorem (2.1.9) in above equation, we get

$$\Omega = \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(\lambda + \mu n) (-n)_{mr} (\alpha)_{n(\beta+1)} [(a_p)]_r x^r t^n}{\alpha (\alpha + 1)_{n\beta} (1 + \alpha + n\beta)_{mr} [(b_q)]_r r! n!} \quad (3.2.3)$$

Now applying summation identity (2.1.10) and then simplifying further, we get

$$\Omega = \sum_{r=0}^{\infty} \frac{(\alpha)_{m(\beta+1)r} [(a_p)]_r x^r (-t)^{mr}}{\alpha (\alpha + 1)_{m(\beta+1)r} [(b_q)]_r r!} \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} (\lambda + \mu n + \mu mr)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \cdot \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \quad (3.2.4)$$

Now using first modified Gould's identity (2.1.7) and then (3.1.2), we get

$$\Omega = (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \frac{(\alpha)_{m(\beta+1)r} [(a_p)]_r x^r (-t)^{mr}}{\alpha (\alpha + 1)_{m(\beta+1)r} [(b_q)]_r r!} (1 + \zeta)^{m(\beta+1)r} \cdot \left[(\lambda + \mu mr) + \frac{\mu \zeta \{\alpha + (\beta + 1)mr\}}{(1 - \beta \zeta)} \right] \quad (3.2.5)$$

Simplifying it further, we get

$$\Omega = (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \frac{\left(\frac{\alpha}{m(\beta+1)} \right)_r [(a_p)]_r x^r (-\zeta)^{mr}}{\alpha \left(\frac{\alpha}{m(\beta+1)} + 1 \right)_r [(b_q)]_r r!} \cdot \left[\lambda \frac{\left(\frac{\lambda}{\mu m} + 1 \right)_r}{\left(\frac{\lambda}{\mu m} \right)_r} + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\alpha}{m(\beta+1)} + 1 \right)_r}{\left(\frac{\alpha}{m(\beta+1)} \right)_r} \right] \quad (3.2.6)$$

After solving it further, we get the result (3.2.1) in the form of sum of two generalized hypergeometric functions of one variable.

Second Generating Relation:

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n = (1 + \zeta)^\alpha .$$

$$\cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda}{\mu(\beta+1)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda}{\mu(\beta+1)}, (b_q); \end{matrix} x(-\zeta)^m \right] + \frac{\mu\zeta}{(1 - \beta\zeta)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (3.2.7)$$

where

$$\zeta = t(1 + \zeta)^{1+\beta}; \zeta(0) = 0.$$

provided that involved series on both sides are absolutely convergent.

Here Pathan's generalized hypergeometric polynomials of one variable $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ are defined by equation (2.2.8).

Independent Demonstration:

Using the definition (2.2.8) of $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ and then the power series form of ${}_{p+m+1}F_{q+m+1}[x]$ in left hand side of equation (3.2.7), using Gauss's multiplication theorem (2.1.9) and result (2.1.21), we get

$$\Omega^* = \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n =$$

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{[\lambda + \mu n + \mu(\beta + 1)r - \mu mr] (-n)_{mr} \Gamma\{\alpha + n(\beta + 1)\} [(a_p)]_r x^r t^n}{\Gamma\{\alpha + n\beta + 1 + mr\} [(b_q)]_r r! n!} \quad (3.2.8)$$

Now applying summation identity (2.1.10) in above equation then simplifying further, we get

$$\Omega^* = \sum_{r=0}^{\infty} \frac{[(a_p)]_r x^r (-t)^{mr}}{[\alpha + m(\beta + 1)r] [(b_q)]_r r!} \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} (\lambda + \mu n + \mu(\beta + 1)r)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \cdot \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \quad (3.2.9)$$

Now using second modified Gould's identity (2.1.8), we get

$$\Omega^* = (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \frac{[(a_p)]_r x^r (-t)^{mr} (1 + \zeta)^{m(\beta+1)r}}{[(b_q)]_r r!} \cdot \left[\frac{\lambda + \mu(\beta + 1)r}{\alpha + m(\beta + 1)r} + \frac{\mu\zeta}{(1 - \beta\zeta)} \right] \quad (3.2.10)$$

Now using equation (3.1.1) in above equation and summing it up into hypergeometric form further, we get the desired result (3.2.7).

3.3 Known Applications of First Generating Relation (3.2.1)

(i). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ in equation (3.2.1) and after simplifying, we get

$$\begin{aligned} \sum_{n=0}^{\infty} H_n^{(\alpha, \beta)}(x; m) t^n &= (1 + \zeta)^\alpha {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta)^m \right] \left\{ \frac{\zeta(1 + \beta)}{(1 - \beta \zeta)} + 1 \right\} \\ &= \frac{(1 + \zeta)^{\alpha+1}}{(1 - \beta \zeta)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta)^m \right] \end{aligned} \quad (3.3.1)$$

which is the result of Srivastava [106, p.975; 107, p.233, eq.(12)]. Here ζ being given by equations (2.1.3), (2.1.4) and $H_n^{(\alpha, \beta)}(x; m)$ is given by equation (2.2.2).

(ii). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ and $m = 1$ in equation (3.2.1) and using the definition of $H_n^{(\alpha, \beta)}(x; 1)$ and after simplification, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)n}{n} {}_{p+1}F_{q+1} \left[\begin{matrix} -n, (a_p) \\ 1 + \alpha + n\beta, (b_q); \end{matrix} x \right] t^n \\ = \frac{(1 + \zeta)^{\alpha+1}}{(1 - \beta \zeta)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta) \right] \end{aligned} \quad (3.3.2)$$

which is the result of Srivastava [105, p.591, eq.(9); see also 59, p.1186]. ζ is given by equations (2.1.3) and (2.1.4).

(iii). Putting $\lambda = 1, \mu = \frac{1}{2\alpha}, \beta = \frac{-1}{2}$ and $m = 1$ in equation (3.2.1), using the definition (2.2.2) of $H_n^{(\alpha, -\frac{1}{2})}(x; 1) = f_n^{(\alpha)}(x)$ and replacing ζ by U , we get

$$\begin{aligned} \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n &= \sum_{n=0}^{\infty} \binom{\alpha + \frac{n}{2}}{n} {}_{p+1}F_{q+1} \left[\begin{matrix} -n, (a_p) \\ 1 + \alpha - \frac{n}{2}, (b_q); \end{matrix} x \right] t^n \\ &= \frac{[1 + U]^{\alpha+1}}{(1 + \frac{1}{2}U)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} -xU \right] \end{aligned} \quad (3.3.3)$$

where

$$U = \frac{t}{2} \{t + \sqrt{t^2 + 4}\}$$

which is the known result of Brown [7, p.264, eq.(7)] and $f_n^{(\alpha)}(x)$ is Brown's generalized hypergeometric polynomial [117, p.358, eq.(7.2.4)].

(iv). Putting $\alpha = \nu - 1, \beta = 0, \zeta = \frac{t}{1-t}$ and replacing q by $(q - m)$ in equation (3.3.1) and after simplifying, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\nu + n - 1}{n}_{p+m} F_q \left[\begin{matrix} \Delta(m; -n), (a_p) ; \\ \Delta(m; \nu), (b_{q-m}) ; \end{matrix} x \right] t^n \\ &= (1-t)^{-\nu} {}_p F_{q-m} \left[\begin{matrix} (a_p) ; \\ (b_{q-m}) ; \end{matrix} x \left(\frac{t}{t-1} \right)^m \right] \end{aligned} \quad (3.3.4)$$

Now replacing p to $p + m$ in equation (3.3.4), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\nu + n - 1}{n}_{p+2m} F_q \left[\begin{matrix} \Delta(m; -n), a_1, \dots, a_p, a_{p+1}, \dots, a_{p+m} ; \\ \Delta(m; \nu), b_1, \dots, b_{q-m} ; \end{matrix} x \right] t^n \\ &= (1-t)^{-\nu} {}_{p+m} F_{q-m} \left[\begin{matrix} a_1, \dots, a_p, a_{p+1}, \dots, a_{p+m} ; \\ b_1, \dots, b_{q-m} ; \end{matrix} x \left(\frac{t}{t-1} \right)^m \right] \end{aligned} \quad (3.3.5)$$

Further taking

$$a_{p+1} = \frac{\nu}{m}, a_{p+2} = \frac{\nu+1}{m}, \dots, a_{p+m} = \frac{\nu+m-1}{m}$$

in equation (3.3.5), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\nu + n - 1}{n}_{p+m} F_{q-m} \left[\begin{matrix} \Delta(m; -n), a_1, \dots, a_p ; \\ b_1, \dots, b_{q-m} ; \end{matrix} x \right] t^n \\ &= (1-t)^{-\nu} {}_{p+m} F_{q-m} \left[\begin{matrix} \Delta(m; \nu), a_1, \dots, a_p ; \\ b_1, \dots, b_{q-m} ; \end{matrix} x \left(\frac{t}{t-1} \right)^m \right] \end{aligned} \quad (3.3.6)$$

Replacing q by $q + m$ and x by $f(x)$ in equation (3.3.6), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\nu + n - 1}{n}_{p+m} F_q \left[\begin{matrix} \Delta(m; -n), a_1, \dots, a_p ; \\ b_1, \dots, b_q ; \end{matrix} f(x) \right] t^n \\ &= (1-t)^{-\nu} {}_{p+m} F_q \left[\begin{matrix} \Delta(m; \nu), a_1, \dots, a_p ; \\ b_1, \dots, b_q ; \end{matrix} f(x) \left(\frac{t}{t-1} \right)^m \right] \end{aligned} \quad (3.3.7)$$

which is the known result of Srivastava [107, p.233, equation(13)].

3.4 Some Special Cases of Generating Relation

(3.3.2)

(i). Taking $p = 0 = q$ in equation (3.3.2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)n}{n} {}_1F_1 \left[\begin{matrix} -n & ; \\ 1 + \alpha + n\beta & \end{matrix} ; x \right] t^n &= \frac{(1 + \zeta)^{\alpha+1}}{(1 - \beta\zeta)} {}_0F_0 \left[\begin{matrix} - & ; \\ - & \end{matrix} ; x(-\zeta) \right] \\ &= \frac{(1 + \zeta)^{\alpha+1}}{(1 - \beta\zeta)} \exp(-x\zeta) \end{aligned} \quad (3.4.1)$$

which is the known result of Srivastava[105, p.591, eq.(7)] subject to the conditions (2.1.3) and (2.1.4).

Now using the definition of generalized Laguerre polynomials (2.1.12) and solving we get

$$\sum_{n=0}^{\infty} L_n^{(\alpha+n\beta)}(x) t^n = \frac{(1 + \zeta)^{\alpha+1}}{(1 - \beta\zeta)} e^{-x\zeta} \quad (3.4.2)$$

where ζ is given by equations (2.1.3) and (2.1.4). It is the known result of Brown[5, p.822] and Carlitz[14, p.826 ; see also 105, p.590, eq.(4)]. The generating relation (3.4.2) is a unification and generalization of the following two generating relations (3.4.3) and (3.4.4):

Taking $\beta = 0$ in equations (2.1.3), (2.1.4) and (3.4.2), we have

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1 - t)^{-\alpha-1} \exp \left\{ -\frac{xt}{(1 - t)} \right\} \quad (3.4.3)$$

and taking $\beta = -1$ in equations (2.1.3), (2.1.4) and (3.4.2), we have

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x) t^n = (1 + t)^{\alpha} \exp(-xt) \quad (3.4.4)$$

(ii). In equations (2.1.3), (2.1.4) and (3.3.2), putting $p = 1, q = 0$; replacing x by $\frac{1-x}{2}$, β by b and ζ by v , we get

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{\alpha + (b + 1)n}{n} {}_2F_1 \left[\begin{matrix} -n, a_1 & ; & \frac{(1 - x)}{2} \\ 1 + \alpha + nb & \end{matrix} \right] t^n \\ = \frac{(1 + v)^{\alpha+1}}{(1 - bv)} {}_1F_0 \left[\begin{matrix} a_1 & ; & \frac{(1 - x)}{2}(-v) \\ - & \end{matrix} \right] \end{aligned}$$

$$= \frac{(1+v)^{\alpha+1}}{(1-bv)} \left\{ 1 + \frac{(1-x)}{2}v \right\}^{-a_1} \quad (3.4.5)$$

where v is a function of 't', defined implicitly by

$$v = t(1+v)^{(b+1)} \quad ; v(0) = 0 \quad (3.4.6)$$

It is the known Generating relation of Srivastava[105, p.591, eq.(8)]. Putting $a_1 = 1 + \alpha + \beta$ in equation (3.4.5) and using the definition (2.1.15) of Jacobi polynomials, we get

$$\sum_{n=0}^{\infty} P_n^{(\alpha+bn, \beta-(b+1)n)}(x) t^n = \frac{(1+v)^{\alpha+1}}{(1-bv)} \left\{ 1 + \frac{(1-x)}{2}v \right\}^{-(1+\alpha+\beta)} \quad (3.4.7)$$

where v is given by equation (3.4.6). This is the known result of Srivastava [105, p.594, eq.(22); 120, p.748 ; see also 11, p.A654].

Taking $b = 0$ in equations (3.4.6) and (3.4.7), we get the following generating relation of E.Feldheim, recorded in the monograph of Srivastava-Manocha [117, p.90, Q.15 (second equation); see also 105, p.594, eq.(23)].

$$\sum_{n=0}^{\infty} P_n^{(\alpha, \beta-n)}(x) t^n = (1-t)^{\beta} \left\{ 1 - \frac{(1+x)t}{2} \right\}^{-(1+\alpha+\beta)} \quad (3.4.8)$$

(iii). Putting $p = 1, q = 0, a_1 = \lambda, \alpha = a, \beta = b$ and replacing ζ by v in equations (2.1.3), (2.1.4) and (3.3.2), we get

$$\sum_{n=0}^{\infty} \binom{a+(b+1)n}{n} {}_2F_1 \left[\begin{matrix} -n, \lambda \\ 1+a+nb \end{matrix} ; x \right] t^n = \frac{(1+v)^{a+1}}{(1-bv)} (1+xv)^{-\lambda} \quad (3.4.9)$$

where

$$v = t(1+v)^{(b+1)} \quad ; v(0) = 0$$

Replacing x by $\frac{2}{(1-x)}$ and λ by $(-\alpha)$ and then a by $(-\alpha - \beta - 1)$ and using the definition (2.1.16) of Jacobi Polynomials, we get the following result

$$\sum_{n=0}^{\infty} \left(\frac{2}{1-x} \right)^n P_n^{(\alpha-n, \beta-bn-n)}(x) t^n = \frac{(1+v)^{-\alpha-\beta}}{(1-bv)} \left(1 + \frac{2}{(1-x)}v \right)^{\alpha} \quad (3.4.10)$$

where

$$v = t(1+v)^{(b+1)} \quad ; v(0) = 0$$

Replacing t by $t \frac{(1-x)}{2}$; v by w , we get

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n, \beta-bn-n)}(x) t^n = \frac{(1+w)^{-\alpha-\beta}}{(1-bw)} \left(1 + \frac{2w}{(1-x)} \right)^{\alpha} \quad (3.4.11)$$

where

$$w(x, t) = t \frac{(1-x)}{2} (1+w)^{(b+1)} ; w(x, 0) = 0 \quad (3.4.12)$$

which is known result of Srivastava [105, p.593 , eq.(16); see also 120, p.748].

Taking $b = -1$ in equations (3.4.6) and (3.4.7); (3.4.11) and (3.4.12), we obtain another result of E.Feldheim, recorded in the monograph of Srivastava-Manocha [117, p.90, Q.15 (first equation); see also 105, p.593, eq.(19)]

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n, \beta)}(x) t^n = (1+t)^\alpha \left(1 - \frac{t(x-1)}{2}\right)^{-\alpha-\beta-1} \quad (3.4.13)$$

And taking $b = 0$ in equations (3.4.11) and (3.4.12), we obtain

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n, \beta-n)}(x) t^n = \left(1 + \frac{t(x+1)}{2}\right)^\alpha \left(1 + \frac{t(x-1)}{2}\right)^\beta \quad (3.4.14)$$

which is the result of Milch [79] and also recorded in the monograph of Srivastava-Manocha [117, p.82, (1.11.2)].

(iv). Putting $p = 2, q = 1, \beta = b, a_1 = 1 + \alpha + \gamma, a_2 = \mu, b_1 = \sigma$ and replacing ζ by v in equations (2.1.3), (2.1.4) and (3.3.2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \binom{\alpha + (b+1)n}{n} {}_3F_2 \left[\begin{matrix} -n, 1 + \alpha + \gamma, \mu; \\ 1 + \alpha + nb, \sigma; \end{matrix} x \right] t^n \\ = \frac{(1+v)^{\alpha+1}}{(1-bv)} {}_2F_1 \left[\begin{matrix} 1 + \alpha + \gamma, \mu; \\ \sigma; \end{matrix} x(-v) \right] \end{aligned} \quad (3.4.15)$$

$v = t(1+v)^{(b+1)}; v(0) = 0$

Now using the definition (2.1.20) of Generalized Rice polynomials of Khandekar in above equation (3.4.15), we get a well known result of Joshi and Prajapat [57, p.272]:

$$\sum_{n=0}^{\infty} H_n^{(\alpha+bn, \gamma-bn-n)}[\mu, \sigma, x] t^n = \frac{(1+v)^{\alpha+1}}{(1-bv)} {}_2F_1 \left[\begin{matrix} 1 + \alpha + \gamma, \mu; \\ \sigma; \end{matrix} x(-v) \right] \quad (3.4.16)$$

where

$$v = t(1+v)^{(b+1)}; v(0) = 0.$$

(v). Putting $b = \frac{-1}{2}$ and replacing v by U in equations (3.4.6) and (3.4.7), we get

$$\sum_{n=0}^{\infty} P_n^{(\alpha-\frac{n}{2}, \beta-\frac{n}{2})}(x) t^n = \frac{(1+U)^{\alpha+1}}{(1+\frac{U}{2})} \left\{ 1 + \frac{(1-x)}{2} U \right\}^{-(1+\alpha+\beta)} \quad (3.4.17)$$

$$U = \frac{t\{t + \sqrt{(t^2 + 4)}\}}{2}$$

which is the known result of Brown [5].

3.5 New Applications of Generating Relation

(3.2.1)

The results from (3.5.1) to (3.5.8) are believed to be new in author's knowledge and are not found in the literature of generating relations.

(i). Putting $\beta = 0$ and $\zeta = \Theta = \frac{t}{1-t}$ from equation (2.1.24) in equation (3.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + n)} \binom{\alpha + n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) \\ \Delta(m; 1 + \alpha), (b_q); \end{matrix} ; x \right] t^n = (1 - t)^{-\alpha} \cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m}, 1 + \frac{\lambda}{m\mu}, (a_p); \\ 1 + \frac{\alpha}{m}, \frac{\lambda}{m\mu}, (b_q); \end{matrix} ; x \left(\frac{-t}{1-t} \right)^m \right] + \frac{\mu t}{(1-t)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} ; x \left(\frac{-t}{1-t} \right)^m \right] \right\} \quad (3.5.1)$$

(ii). Putting $\beta = 1$ and $\zeta = \Lambda = \frac{1-2t-\sqrt{(1-4t)}}{2t}$ from equation (2.1.25) in equation (3.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + 2n)} \binom{\alpha + 2n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) \\ \Delta(m; 1 + \alpha + n), (b_q); \end{matrix} ; x \right] t^n = (1 + \Lambda)^{\alpha} \cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{2m}, 1 + \frac{\lambda}{m\mu}, (a_p); \\ 1 + \frac{\alpha}{2m}, \frac{\lambda}{m\mu}, (b_q); \end{matrix} ; x(-\Lambda)^m \right] + \frac{\mu \Lambda}{(1 - \Lambda)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} ; x(-\Lambda)^m \right] \right\} \quad (3.5.2)$$

(iii). Putting $\beta = -2$ and $\zeta = \Xi = \frac{-1+\sqrt{(1+4t)}}{2}$ from equation (2.1.26) in equation (3.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - n)} \binom{\alpha - n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) \\ \Delta(m; 1 + \alpha - 2n), (b_q); \end{matrix} ; x \right] t^n = (1 + \Xi)^{\alpha} \cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{-\alpha}{m}, 1 + \frac{\lambda}{m\mu}, (a_p); \\ 1 - \frac{\alpha}{m}, \frac{\lambda}{m\mu}, (b_q); \end{matrix} ; x(-\Xi)^m \right] + \frac{\mu \Xi}{(1 + 2\Xi)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} ; x(-\Xi)^m \right] \right\} \quad (3.5.3)$$

(iv). Putting $\beta = -3$ and $\zeta = \Upsilon$ from equation (2.1.27) in equation (3.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - 2n)} \binom{\alpha - 2n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - 3n), (b_q) & ; & \end{matrix} \right] t^n = (1 + \Upsilon)^\alpha.$$

$$\cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{-\alpha}{2m}, 1 + \frac{\lambda}{m\mu}, (a_p); & x(-\Upsilon)^m \\ 1 - \frac{\alpha}{2m}, \frac{\lambda}{m\mu}, (b_q); & \end{matrix} \right] + \frac{\mu \Upsilon}{(1 + 3\Upsilon)} {}_pF_q \left[\begin{matrix} (a_p); & x(-\Upsilon)^m \\ (b_q); & \end{matrix} \right] \right\} \quad (3.5.4)$$

(v). Putting $\beta = \frac{-1}{2}$ and $\zeta = U$ from equation (2.1.28) in equation (3.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{2}n)} \binom{\alpha + \frac{1}{2}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - \frac{1}{2}n), (b_q) & ; & \end{matrix} \right] t^n = (1 + U)^\alpha.$$

$$\cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{2\alpha}{m}, 1 + \frac{\lambda}{m\mu}, (a_p); & x(-U)^m \\ 1 + \frac{2\alpha}{m}, \frac{\lambda}{m\mu}, (b_q); & \end{matrix} \right] + \frac{\mu U}{(1 + \frac{1}{2}U)} {}_pF_q \left[\begin{matrix} (a_p); & x(-U)^m \\ (b_q); & \end{matrix} \right] \right\} \quad (3.5.5)$$

(vi). Putting $\beta = \frac{-3}{2}$ and $\zeta = \Psi$ from equation (2.1.29) in equation (3.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - \frac{1}{2}n)} \binom{\alpha - \frac{1}{2}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - \frac{3}{2}n), (b_q) & ; & \end{matrix} \right] t^n = (1 + \Psi)^\alpha.$$

$$\cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{-2\alpha}{m}, 1 + \frac{\lambda}{m\mu}, (a_p); & x(-\Psi)^m \\ 1 - \frac{2\alpha}{m}, \frac{\lambda}{m\mu}, (b_q); & \end{matrix} \right] + \frac{\mu \Psi}{(1 + \frac{3}{2}\Psi)} {}_pF_q \left[\begin{matrix} (a_p); & x(-\Psi)^m \\ (b_q); & \end{matrix} \right] \right\} \quad (3.5.6)$$

(vii). Putting $\beta = \frac{-1}{3}$ and $\zeta = \Pi$ from equation (2.1.30) in equation (3.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{2}{3}n)} \binom{\alpha + \frac{2}{3}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - \frac{1}{3}n), (b_q) & ; & \end{matrix} \right] t^n = (1 + \Pi)^\alpha.$$

$$\cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{3\alpha}{2m}, 1 + \frac{\lambda}{m\mu}, (a_p); & x(-\Pi)^m \\ 1 + \frac{3\alpha}{2m}, \frac{\lambda}{m\mu}, (b_q); & \end{matrix} \right] + \frac{\mu \Pi}{(1 + \frac{1}{3}\Pi)} {}_pF_q \left[\begin{matrix} (a_p); & x(-\Pi)^m \\ (b_q); & \end{matrix} \right] \right\} \quad (3.5.7)$$

(viii). Putting $\beta = \frac{-2}{3}$ and $\zeta = \Phi$ from equation (2.1.31) in equation (3.2.1), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{3}n)} \binom{\alpha + \frac{1}{3}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - \frac{2}{3}n), (b_q) & ; & \end{matrix} \right] t^n = (1 + \Phi)^\alpha.$$

$$\cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{3\alpha}{m}, 1 + \frac{\lambda}{m\mu}, (a_p); \\ 1 + \frac{3\alpha}{m}, \frac{\lambda}{m\mu}, (b_q); \end{matrix} x(-\Phi)^m \right] + \frac{\mu \Phi}{(1 + \frac{2}{3} \Phi)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\Phi)^m \right] \right\} \quad (3.5.8)$$

3.6 New Applications of Generating Relation (3.2.7)

The following Generating relations of this section are new in the author's knowledge and are not available in the literature of Generating relations.

(i). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ in equation (3.2.7) and after simplifying, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)n}{n} {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\alpha + (\beta + 1)n}{(\beta + 1)(\beta + 1 - m)}, (a_p); \\ \Delta(m; 1 + \alpha + \beta n), \frac{\alpha + (\beta + 1)n}{(\beta + 1)(\beta + 1 - m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1 + \zeta)^\alpha \cdot \left\{ {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta + 1)}, 1 + \frac{\alpha}{(\beta + 1)^2}, (a_p); \\ 1 + \frac{\alpha}{m(\beta + 1)}, \frac{\alpha}{(\beta + 1)^2}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\ & \quad \left. + \frac{(\beta + 1)\zeta}{(1 - \beta\zeta)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \end{aligned} \quad (3.6.1)$$

where ζ is given by equations (2.1.3) and (2.1.4).

(ii). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}, m = 1$ in equation (3.2.7) and using the definition of $\mathcal{B}_n^{(\alpha, \beta)}(x; 1, 1, \frac{\beta+1}{\alpha})$ from equation (2.2.8), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)n}{n} {}_{p+2}F_{q+2} \left[\begin{matrix} -n, 1 + \frac{\alpha + (\beta + 1)n}{(\beta + 1)\beta}, (a_p); \\ 1 + \alpha + \beta n, \frac{\alpha + (\beta + 1)n}{(\beta + 1)\beta}, (b_q); \end{matrix} x \right] t^n = (1 + \zeta)^\alpha \cdot \\ & \cdot \left\{ {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{(\beta + 1)}, 1 + \frac{\alpha}{(\beta + 1)^2}, (a_p); \\ 1 + \frac{\alpha}{(\beta + 1)}, \frac{\alpha}{(\beta + 1)^2}, (b_q); \end{matrix} x(-\zeta) \right] + \frac{(\beta + 1)\zeta}{(1 - \beta\zeta)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta) \right] \right\} \end{aligned} \quad (3.6.2)$$

where ζ is given by equations (2.1.3) and (2.1.4).

(iii). Putting $\lambda = 1, \mu = \frac{1}{2\alpha}, \beta = \frac{-1}{2}, m = 1$ in equation (3.2.7), using the definition of $\mathcal{B}_n^{(\alpha, -\frac{1}{2})}(x; 1, 1, \frac{1}{2\alpha})$ and replacing ζ by U from equation (2.1.28), we get

$$\sum_{n=0}^{\infty} \binom{\alpha + \frac{n}{2}}{n} {}_{p+2}F_{q+2} \left[\begin{matrix} -n, 1 - 4\alpha - 2n, (a_p) & ; \\ 1 + \alpha - \frac{n}{2}, -4\alpha - 2n, (b_q) & ; \end{matrix} x \right] t^n = (1 + U)^\alpha.$$

$$\cdot \left\{ {}_{p+2}F_{q+2} \left[\begin{matrix} 2\alpha, 1+4\alpha, (a_p); \\ 1+2\alpha, 4\alpha, (b_q); \end{matrix} x(-U) \right] + \frac{U}{(2+U)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-U) \right] \right\} \quad (3.6.3)$$

(iv). Putting $\beta = 0$ and $\zeta = \Theta = \frac{t}{1-t}$ from equation (2.1.24) in equation (3.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + n)} \binom{\alpha + n}{n} {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(1-m)}, (a_p); \\ \Delta(m; 1 + \alpha), \frac{\lambda + \mu n}{\mu(1-m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1-t)^{-\alpha} \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m}, 1 + \frac{\lambda}{\mu}, (a_p); \\ 1 + \frac{\alpha}{m}, \frac{\lambda}{\mu}, (b_q); \end{matrix} x \left(\frac{t}{t-1} \right)^m \right] \right. \\ & \quad \left. + \mu \left(\frac{t}{1-t} \right) {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x \left(\frac{t}{t-1} \right)^m \right] \right\} \quad (3.6.4) \end{aligned}$$

(v). Putting $\beta = 1$ and $\zeta = \Lambda = \frac{1-2t-\sqrt{(1-4t)}}{2t}$ from equation (2.1.25) in equation (3.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + 2n)} \binom{\alpha + 2n}{n} {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(2-m)}, (a_p); \\ \Delta(m; 1 + \alpha + n), \frac{\lambda + \mu n}{\mu(2-m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1+\Lambda)^{\alpha} \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{2m}, 1 + \frac{\lambda}{2\mu}, (a_p); \\ 1 + \frac{\alpha}{2m}, \frac{\lambda}{2\mu}, (b_q); \end{matrix} x(-\Lambda)^m \right] \right. \\ & \quad \left. + \frac{\mu\Lambda}{(1-\Lambda)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\Lambda)^m \right] \right\} \quad (3.6.5) \end{aligned}$$

(vi). Putting $\beta = -2$ and $\zeta = \Xi = \frac{-1+\sqrt{(1+4t)}}{2}$ from equation (2.1.26) in equation (3.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - n)} \binom{\alpha - n}{n} {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 - \frac{(\lambda + \mu n)}{\mu(1+m)}, (a_p); \\ \Delta(m; 1 + \alpha - 2n), -\frac{(\lambda + \mu n)}{\mu(1+m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1+\Xi)^{\alpha} \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{-\alpha}{m}, 1 - \frac{\lambda}{\mu}, (a_p); \\ 1 - \frac{\alpha}{m}, \frac{-\lambda}{\mu}, (b_q); \end{matrix} x(-\Xi)^m \right] \right. \\ & \quad \left. + \frac{\mu\Xi}{(1+2\Xi)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\Xi)^m \right] \right\} \quad (3.6.6) \end{aligned}$$

(vii). Putting $\beta = -3$ and $\zeta = \Upsilon$ from equation (2.1.27) in equation (3.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - 2n)} \binom{\alpha - 2n}{n}_{p+m+1} F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 - \frac{(\lambda + \mu n)}{\mu(2+m)}, (a_p); \\ \Delta(m; 1 + \alpha - 3n), -\frac{(\lambda + \mu n)}{\mu(2+m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1 + \Upsilon)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{-\alpha}{2m}, 1 - \frac{\lambda}{2\mu}, (a_p); \\ 1 - \frac{\alpha}{2m}, \frac{-\lambda}{2\mu}, (b_q); \end{matrix} x(-\Upsilon)^m \right] \right. \\ & \quad \left. + \frac{\mu \Upsilon}{(1 + 3\Upsilon)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\Upsilon)^m \right] \right\} \end{aligned} \quad (3.6.7)$$

(viii). Putting $\beta = \frac{-1}{2}$ and $\zeta = U$ from equation (2.1.28) in equation (3.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{2}n)} \binom{\alpha + \frac{1}{2}n}{n}_{p+m+1} F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{(\lambda + \mu n)}{\mu(\frac{1}{2}-m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{1}{2}n), \frac{(\lambda + \mu n)}{\mu(\frac{1}{2}-m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1 + U)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{2\alpha}{m}, 1 + \frac{2\lambda}{\mu}, (a_p); \\ 1 + \frac{2\alpha}{m}, \frac{2\lambda}{\mu}, (b_q); \end{matrix} x(-U)^m \right] \right. \\ & \quad \left. + \frac{\mu U}{(1 + \frac{1}{2}U)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-U)^m \right] \right\} \end{aligned} \quad (3.6.8)$$

(ix). Putting $\beta = \frac{-3}{2}$ and $\zeta = \Psi$ from equation (2.1.29) in equation (3.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - \frac{1}{2}n)} \binom{\alpha - \frac{1}{2}n}{n}_{p+m+1} F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 - \frac{(\lambda + \mu n)}{\mu(\frac{1}{2}+m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{3}{2}n), -\frac{(\lambda + \mu n)}{\mu(\frac{1}{2}+m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1 + \Psi)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{-2\alpha}{m}, 1 - \frac{2\lambda}{\mu}, (a_p); \\ 1 - \frac{2\alpha}{m}, \frac{-2\lambda}{\mu}, (b_q); \end{matrix} x(-\Psi)^m \right] \right. \\ & \quad \left. + \frac{\mu \Psi}{(1 + \frac{3}{2}\Psi)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\Psi)^m \right] \right\} \end{aligned} \quad (3.6.9)$$

(x). Putting $\beta = \frac{-1}{3}$ and $\zeta = \Pi$ from equation (2.1.30) in equation (3.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{2}{3}n)} \binom{\alpha + \frac{2}{3}n}{n}_{p+m+1} F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{(\lambda + \mu n)}{\mu(\frac{2}{3}-m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{1}{3}n), \frac{(\lambda + \mu n)}{\mu(\frac{2}{3}-m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1 + \Pi)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{3\alpha}{2m}, 1 + \frac{3\lambda}{2\mu}, (a_p); \\ 1 + \frac{3\alpha}{2m}, \frac{3\lambda}{2\mu}, (b_q); \end{matrix} x(-\Pi)^m \right] \right. \end{aligned}$$

$$+ \frac{\mu \Pi}{(1 + \frac{1}{3} \Pi)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\Pi)^m \right] \Bigg\} \quad (3.6.10)$$

(xi). Putting $\beta = \frac{-2}{3}$ and $\zeta = \Phi$ from equation (2.1.31) in equation (3.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{3}n)} \binom{\alpha + \frac{1}{3}n}{n} {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{(\lambda + \mu n)}{\mu(\frac{1}{3}-m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{2}{3}n), \frac{(\lambda + \mu n)}{\mu(\frac{1}{3}-m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1 + \Phi)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{3\alpha}{m}, 1 + \frac{3\lambda}{\mu}, (a_p); \\ 1 + \frac{3\alpha}{m}, \frac{3\lambda}{\mu}, (b_q); \end{matrix} x(-\Phi)^m \right] \right. \\ & \quad \left. + \frac{\mu \Phi}{(1 + \frac{2}{3} \Phi)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\Phi)^m \right] \right\} \quad (3.6.11) \end{aligned}$$

Making suitable adjustments of parameters and variables in all generating relations of sections 3.5 and 3.6, we can also obtain a number of new generating relations involving restricted generalized Laguerre polynomials, restricted Jacobi polynomials, restricted generalized Rice polynomials of Khandekar and other orthogonal polynomials.

Chapter 4

Certain Hypergeometric Generating Relations via Gould's Identity (Part-III)

4.1 Introduction and Preliminaries:

This chapter brings into light some another hypergeometric generating relations associated with two hypergeometric polynomials of one variable $H_n^{(\alpha, \beta)}(x; m)$ and $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$. Furthermore, their independent demonstrations are given using Gould's identity. A number of known and new generating relations are deduced as particular cases. Under constraints, further generalizations of two main hypergeometric generating relations have also been given for two generalized polynomials $S_n^{(\alpha, \beta)}(x; m)$ and $T_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$.

Some useful Pochhammer's relations

$$\mu(\beta+1)r + \frac{\mu\zeta}{(1-\beta\zeta)}\{\alpha + m(\beta+1)r\} = \frac{\mu\zeta\alpha}{(1-\beta\zeta)} \frac{\left(\frac{\zeta\alpha}{(\beta+1)(1+m\zeta-\beta\zeta)} + 1\right)_r}{\left(\frac{\zeta\alpha}{(\beta+1)(1+m\zeta-\beta\zeta)}\right)_r} \quad (4.1.1)$$

$$\mu m r + \frac{\mu\zeta\{\alpha + (\beta+1)m r\}}{(1-\beta\zeta)} = \frac{\mu\zeta}{(1-\beta\zeta)} \frac{\left(\frac{\alpha\zeta}{m(1+\zeta)} + 1\right)_r}{\left(\frac{\alpha\zeta}{m(1+\zeta)}\right)_r} \quad (4.1.2)$$

where

$$r = 0, 1, 2, 3, \dots$$

4.2 Main Generating Relations

4.2.1 First Generating Relation:

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta+1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n &= (1+\zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p) \\ 1 + \frac{\alpha}{m(\beta+1)}, (b_q) \end{matrix} ; x(-\zeta)^m \right] \right. \\ &\quad \left. + \frac{\mu\zeta}{(1-\beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p), 1 + \frac{\alpha\zeta}{m(\zeta+1)} \\ 1 + \frac{\alpha}{m(\beta+1)}, (b_q), \frac{\alpha\zeta}{m(\zeta+1)} \end{matrix} ; x(-\zeta)^m \right] \right\} \end{aligned} \quad (4.2.1)$$

where

$$\zeta = t(1 + \zeta)^{\beta+1} \quad ; \zeta(0) = 0$$

provided that involved series on both sides are absolutely convergent.

Here Srivastava's generalized hypergeometric polynomials $H_n^{(\alpha, \beta)}(x; m)$ [117, p.360, eq.(7.3.3); see also 109, pp.331 – 332] are given by equation (2.2.2).

Independent Demonstration:

Using definition (2.2.2) of $H_n^{(\alpha, \beta)}(x; m)$ and then the power series form of ${}_{p+m}F_{q+m}[x]$ in left hand side of eq.(4.2.1), we get

$$\begin{aligned} \Omega &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n = \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \binom{\alpha + (\beta + 1)n}{n} \\ &\quad \cdot {}_{p+m}F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) \\ \Delta(m; 1 + \alpha + n\beta), (b_q) \end{matrix} ; x \right] t^n \\ &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n) \Gamma\{\alpha + (\beta + 1)n + 1\}}{\{\alpha + (\beta + 1)n\} \Gamma(n + 1) \Gamma\{\alpha + \beta n + 1\}} \\ &\quad \cdot \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{\prod_{j=1}^m \left(\frac{-n + j - 1}{m} \right)_r [(a_p)]_r x^r t^n}{\prod_{j=1}^m \left(\frac{1 + \alpha + n\beta + j - 1}{m} \right)_r [(b_q)]_r r!} \end{aligned} \quad (4.2.2)$$

Using Gauss's multiplication theorem (2.1.9) in above equation, we get

$$\Omega = \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(\lambda + \mu n) (-n)_{mr} (\alpha)_{n(\beta+1)} [(a_p)]_r x^r t^n}{(\alpha + 1)_{n\beta} (1 + \alpha + n\beta)_{mr} [(b_q)]_r r! n!} \quad (4.2.3)$$

Now applying summation identity (2.1.10) and then simplifying further, we get

$$\begin{aligned} \Omega &= \sum_{r=0}^{\infty} \frac{(\alpha)_{m(\beta+1)r} [(a_p)]_r x^r (-t)^{mr}}{(\alpha + 1)_{m(\beta+1)r} [(b_q)]_r r!} \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} (\lambda + \mu n + \mu mr)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \\ &\quad \cdot \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \end{aligned} \quad (4.2.4)$$

Now using first modified Gould's identity (2.1.7), we get

$$\begin{aligned} \Omega &= (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \frac{(\alpha)_{m(\beta+1)r} [(a_p)]_r x^r (-t)^{mr}}{(\alpha + 1)_{m(\beta+1)r} [(b_q)]_r r!} (1 + \zeta)^{m(\beta+1)r} \\ &\quad \cdot \left[\lambda + \left(\mu m r + \frac{\mu \zeta \{\alpha + (\beta + 1)mr\}}{(1 - \beta \zeta)} \right) \right] \end{aligned} \quad (4.2.5)$$

Simplifying it further and using equations (2.1.3), (2.1.4) and (4.1.2), we get

$$\Omega = \lambda (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \frac{\left(\frac{\alpha}{m(\beta+1)} \right)_r [(a_p)]_r x^r (-\zeta)^{mr}}{\left(\frac{\alpha}{m(\beta+1)} + 1 \right)_r [(b_q)]_r r!} +$$

$$(1 + \zeta)^\alpha \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \sum_{r=0}^{\infty} \frac{\left(\frac{\alpha}{m(\beta+1)}\right)_r [(a_p)]_r \left(\frac{\alpha \zeta}{m(\zeta+1)} + 1\right)_r x^r (-\zeta)^{mr}}{\left(\frac{\alpha}{m(\beta+1)} + 1\right)_r [(b_q)]_r \left(\frac{\alpha \zeta}{m(\zeta+1)}\right)_r r!} \quad (4.2.6)$$

after solving it further, we get the desired result (4.2.1).

4.2.2 Second Generating Relation:

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\ & \quad \left. + \frac{\mu \zeta}{(1 - \beta \zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} (a_p), \frac{\alpha}{m(\beta+1)}, \frac{\zeta \alpha}{(\beta+1)(1+m\zeta-\beta\zeta)} + 1; \\ (b_q), \frac{\alpha}{m(\beta+1)} + 1, \frac{\zeta \alpha}{(\beta+1)(1+m\zeta-\beta\zeta)}; \end{matrix} x(-\zeta)^m \right] \right\} \quad (4.2.7) \end{aligned}$$

where

$$\zeta = t(1 + \zeta)^{\beta+1} \quad ; \zeta(0) = 0$$

provided that involved series on both sides are absolutely convergent.

Here Pathan's generalized hypergeometric polynomials of one variable $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ are given by (2.2.8).

Independent Demonstration:

Using the definition (2.2.8) of $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ and then the power series form of ${}_{p+m+1}F_{q+m+1}[x]$ in left hand side of equation (4.2.7), using Gauss's multiplication theorem (2.1.9) and result (2.1.21), we get

$$\begin{aligned} \Omega_1 &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n = \\ & \sum_{n=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{[\lambda + \mu n + \mu(\beta + 1)r - \mu mr] (-n)_{mr} \Gamma\{\alpha + n(\beta + 1)\} [(a_p)]_r x^r t^n}{\Gamma\{\alpha + n\beta + 1 + mr\} [(b_q)]_r r! n!} \quad (4.2.8) \end{aligned}$$

Now applying summation identity (2.1.10) in above equation then simplifying further, we get

$$\Omega_1 = \sum_{r=0}^{\infty} \frac{[(a_p)]_r x^r (-t)^{mr}}{[\alpha + m(\beta + 1)r] [(b_q)]_r r!} \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} (\lambda + \mu n + \mu(\beta + 1)r)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}}.$$

$$\cdot \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \quad (4.2.9)$$

Now using second modified Gould's identity (2.1.8), we get

$$\begin{aligned} \Omega_1 = (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \frac{[(a_p)]_r x^r (-t)^{mr} (1 + \zeta)^{m(\beta+1)r}}{\{\alpha + m(\beta + 1)r\} [(b_q)]_r r!} \\ \cdot \left[\lambda + \left(\mu(\beta + 1)r + \frac{\mu\zeta}{(1 - \beta\zeta)} \{\alpha + m(\beta + 1)r\} \right) \right] \end{aligned} \quad (4.2.10)$$

Now using equations (4.1.1) and (2.1.23) in above equation and summing it up into hypergeometric form further, we get the desired result (4.2.7).

4.3 Known Applications of First Generating Relation (4.2.1)

(i). Putting $\lambda = 1, \mu = 0$ in equation (4.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\alpha}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n \\ &= \sum_{n=0}^{\infty} \frac{\alpha}{\{\alpha + (\beta + 1)n\}} \binom{\alpha + (\beta + 1)n}{n} {}_{p+m}F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) \\ \Delta(m; 1 + \alpha + n\beta), (b_q) \end{matrix} ; x \right] t^n \\ &= (1 + \zeta)^\alpha {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, (b_q); \end{matrix} x(-\zeta)^m \right] \\ & \quad \zeta = t(1 + \zeta)^{\beta+1} \quad ; \zeta(0) = 0 \end{aligned} \quad (4.3.1)$$

which is the more simplified form of a result of Srivastava [117, p.366(7.3.19)].

(ii). Putting $\lambda = 1, \mu = 0$ and $m = 1$ in equation (4.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\alpha}{\{\alpha + (\beta + 1)n\}} \binom{\alpha + (\beta + 1)n}{n} {}_{p+1}F_{q+1} \left[\begin{matrix} -n, (a_p) \\ 1 + \alpha + n\beta, (b_q) \end{matrix} ; x \right] t^n \\ &= (1 + \zeta)^\alpha {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{(\beta+1)}, (a_p); \\ 1 + \frac{\alpha}{(\beta+1)}, (b_q); \end{matrix} -x\zeta \right] \\ & \quad \zeta = t(1 + \zeta)^{\beta+1} \quad ; \zeta(0) = 0 \end{aligned} \quad (4.3.2)$$

which is the result of Brown[6; see also 126, p.407, eq.(2.3)].

(iii). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ in equation (4.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; \\ \Delta(m; 1 + \alpha + n\beta), (b_q); & x \end{matrix} \right] t^n \\ &= (1 + \zeta)^\alpha {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p) & ; \\ 1 + \frac{\alpha}{m(\beta+1)}, (b_q); & x(-\zeta)^m \end{matrix} \right] + \frac{(\beta+1)\zeta(1+\zeta)^\alpha}{(1-\beta\zeta)} \\ & \quad \cdot {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p), 1 + \frac{\alpha\zeta}{m(\zeta+1)}; & \\ 1 + \frac{\alpha}{m(\beta+1)}, (b_q), \frac{\alpha\zeta}{m(\zeta+1)}; & x(-\zeta)^m \end{matrix} \right] \end{aligned}$$

After further simplification, we have

$$\begin{aligned} \sum_{n=0}^{\infty} H_n^{(\alpha, \beta)}(x; m) t^n &= \frac{(1 + \zeta)^{(\alpha+1)}}{(1 - \beta\zeta)} {}_pF_q \left[\begin{matrix} (a_p) & ; \\ (b_q) & ; \end{matrix} \right] x(-\zeta)^m \\ \zeta &= t(1 + \zeta)^{\beta+1} \quad ; \zeta(0) = 0 \end{aligned} \quad (4.3.3)$$

which is the known result of Srivastava [106, p.975; see also 107, p.233, eq.(12)].

(iv). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ and $m = 1$ in equation (4.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n}_{p+1} F_{q+1} \left[\begin{matrix} -n, (a_p) & ; \\ 1 + \alpha + n\beta, (b_q); & x \end{matrix} \right] t^n \\ &= (1 + \zeta)^\alpha {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{(\beta+1)}, (a_p); & -x\zeta \\ 1 + \frac{\alpha}{(\beta+1)}, (b_q); & \end{matrix} \right] + \frac{(\beta+1)\zeta(1+\zeta)^\alpha}{(1-\beta\zeta)} \\ & \quad \cdot {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{(\beta+1)}, (a_p), 1 + \frac{\alpha\zeta}{(\zeta+1)}; & \\ 1 + \frac{\alpha}{(\beta+1)}, (b_q), \frac{\alpha\zeta}{(\zeta+1)}; & -x\zeta \end{matrix} \right] \\ &= \frac{(1 + \zeta)^{(\alpha+1)}}{(1 - \beta\zeta)} {}_pF_q \left[\begin{matrix} (a_p) & ; \\ (b_q) & ; \end{matrix} \right] - x\zeta \\ \zeta &= t(1 + \zeta)^{\beta+1} \quad ; \zeta(0) = 0 \end{aligned} \quad (4.3.4)$$

which is the known result of Srivastava[105, p.591, eq.(9); see also 97, p.390].

(v). Putting $\lambda = 1, \mu = 0, \beta = \frac{-1}{2}, m = 1$ and replacing ζ by U in equation (4.2.1), we get

$$\sum_{n=0}^{\infty} \frac{\alpha}{(\alpha + \frac{n}{2})} f_n^{(\alpha)}(x) t^n = \sum_{n=0}^{\infty} \frac{\alpha}{(\alpha + \frac{n}{2})} \binom{\alpha + \frac{n}{2}}{n}_{p+1} F_{q+1} \left[\begin{matrix} -n, (a_p) & ; \\ 1 + \alpha - \frac{n}{2}, (b_q); & x \end{matrix} \right] t^n$$

$$= (1+U)^\alpha {}_{p+1}F_{q+1} \left[\begin{matrix} 2\alpha, (a_p); \\ 1+2\alpha, (b_q); \end{matrix} -xU \right] \quad (4.3.5)$$

where $U = \frac{t}{2}\{t + \sqrt{t^2 + 4}\}$.

This is well known Brown's result [7, p.265, eq.(8)].

(vi). Putting $\lambda = 1, \mu = \frac{1}{2\alpha}, \beta = \frac{-1}{2}, m = 1$ and replacing ζ by U in equation (4.2.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n &= \sum_{n=0}^{\infty} \binom{\alpha + \frac{n}{2}}{n} {}_{p+1}F_{q+1} \left[\begin{matrix} -n, (a_p); \\ 1 + \alpha - \frac{n}{2}, (b_q); \end{matrix} x \right] t^n = (1+U)^\alpha. \\ & {}_{p+1}F_{q+1} \left[\begin{matrix} 2\alpha, (a_p); \\ 1 + 2\alpha, (b_q); \end{matrix} -xU \right] + \frac{U(1+U)^\alpha}{2(1 + \frac{1}{2}U)} {}_{p+2}F_{q+2} \left[\begin{matrix} 2\alpha, (a_p), 1 + \frac{\alpha U}{1+U}; \\ 1 + 2\alpha, (b_q), \frac{\alpha U}{(U+1)}; \end{matrix} -xU \right] \\ &= \frac{(1+U)^{\alpha+1}}{(1 + \frac{1}{2}U)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} -xU \right] \end{aligned} \quad (4.3.6)$$

where

$$U = \frac{t}{2}\{t + \sqrt{t^2 + 4}\}$$

It is also a well known result of Brown [7, p.264, eq.(7)].

4.4 Some Special Cases of Generating Relation(4.3.2)

(i). Taking $p = 0 = q$ in (4.3.2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\alpha}{\{\alpha + (\beta + 1)n\}} \binom{\alpha + (\beta + 1)n}{n} {}_1F_1 \left[\begin{matrix} -n; \\ 1 + \alpha + n\beta; \end{matrix} x \right] t^n \\ = (1 + \zeta)^\alpha {}_1F_1 \left[\begin{matrix} \frac{\alpha}{(\beta+1)}; \\ 1 + \frac{\alpha}{(\beta+1)}; \end{matrix} -x\zeta \right] \end{aligned} \quad (4.4.1)$$

Now using the definition of generalized Laguerre polynomial (2.1.12) and solving we obtain

$$\sum_{n=0}^{\infty} \frac{\alpha}{\{\alpha + (\beta + 1)n\}} L_n^{(\alpha+\beta n)}(x) t^n = (1 + \zeta)^\alpha {}_1F_1 \left[\begin{matrix} \frac{\alpha}{(\beta+1)}; \\ 1 + \frac{\alpha}{(\beta+1)}; \end{matrix} -x\zeta \right] \quad (4.4.2)$$

In equations (4.4.1) and (4.4.2), ζ is given by the equations (2.1.3), (2.1.4). It is the known generating relation of Brown[8, p.27; see also Sharma and Abiodun 94, p.384; Srivastava 110, p.573]

(ii). In equation (4.3.2), putting $p=1, q=0$; replacing x by $(\frac{1-x}{2})$, β by b and ζ by ξ , we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\alpha}{\{\alpha + (b+1)n\}} \binom{\alpha + (b+1)n}{n} {}_2F_1 \left[\begin{matrix} -n, a_1 & ; & \frac{(1-x)}{2} \\ 1 + \alpha + bn & \end{matrix} \right] t^n \\ &= (1+\xi)^\alpha {}_2F_1 \left[\begin{matrix} \frac{\alpha}{b+1}, a_1 & ; & \xi \frac{(x-1)}{2} \\ 1 + \frac{\alpha}{b+1} & \end{matrix} \right] \end{aligned} \quad (4.4.3)$$

where ξ is a function of 't', defined implicitly by

$$\xi = t(1+\xi)^{b+1} \quad ; \xi(0) = 0 \quad (4.4.4)$$

Putting $a_1 = 1 + \alpha + \beta$ in above equation and using eq.(2.1.15), we get

$$\sum_{n=0}^{\infty} \frac{\alpha}{\{\alpha + (b+1)n\}} P_n^{(\alpha+bn, \beta-(b+1)n)}(x) t^n = (1+\xi)^\alpha {}_2F_1 \left[\begin{matrix} \frac{\alpha}{b+1}, 1 + \alpha + \beta & ; & \frac{(x-1)\xi}{2} \\ 1 + \frac{\alpha}{b+1} & \end{matrix} \right] \quad (4.4.5)$$

where ξ is given by equation (4.4.4).

(iii). In equation (4.3.2), putting $p=1, q=0, \alpha=a, \beta=b$, replacing x by $\frac{2}{1-x}$ and ζ by ξ , we find

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{a}{\{a + (b+1)n\}} \binom{a + (b+1)n}{n} {}_2F_1 \left[\begin{matrix} -n, a_1 & ; & \frac{2}{1-x} \\ 1 + a + bn & \end{matrix} \right] t^n = (1+\xi)^a. \\ & \quad {}_2F_1 \left[\begin{matrix} \frac{a}{b+1}, a_1 & ; & \frac{2\xi}{(x-1)} \\ 1 + \frac{a}{b+1} & \end{matrix} \right] \end{aligned} \quad (4.4.6)$$

where

$$\xi = t(1+\xi)^{(b+1)} \quad ; \xi(0) = 0$$

Replacing a_1 by $-\alpha$ and then a by $-\alpha - \beta - 1$, and using eq.(2.1.16), we get the following result

$$\sum_{n=0}^{\infty} \frac{(\alpha + \beta + 1)}{\{\alpha + \beta + 1 - (b+1)n\}} \left(\frac{2}{1-x} \right)^n P_n^{(\alpha-n, \beta-bn-n)}(x) t^n$$

$$= (1 + \xi)^{-(\alpha+\beta+1)} {}_2F_1 \left[\begin{matrix} -\frac{(\alpha+\beta+1)}{b+1}, -\alpha ; \\ 1 - \frac{(\alpha+\beta+1)}{b+1} ; \end{matrix} \frac{2\xi}{(x-1)} \right] \quad (4.4.7)$$

where

$$\xi = t(1 + \xi)^{(b+1)} ; \xi(0) = 0$$

Now replacing t by $t\frac{(1-x)}{2}$ and ξ by w , we find

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\alpha + \beta + 1)}{\{\alpha + \beta + 1 - (b + 1)n\}} P_n^{(\alpha-n, \beta-bn-n)}(x) t^n \\ &= (1 + w)^{-(\alpha+\beta+1)} {}_2F_1 \left[\begin{matrix} -\frac{(\alpha+\beta+1)}{b+1}, -\alpha ; \\ 1 - \frac{(\alpha+\beta+1)}{b+1} ; \end{matrix} \frac{2w}{(x-1)} \right] \end{aligned} \quad (4.4.8)$$

where

$$w(x, t) = t \frac{(1-x)}{2} (1 + w)^{b+1} ; w(x, 0) = 0$$

(iv). putting $p = 2, q = 1, \beta = b, a_1 = 1 + \alpha + \beta, a_2 = \nu$ and $b_1 = \sigma$ and replacing ζ by ξ in equation (4.3.2), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\alpha}{\{\alpha + (b + 1)n\}} \binom{\alpha + (b + 1)n}{n} {}_3F_2 \left[\begin{matrix} -n, 1 + \alpha + \beta, \nu ; \\ 1 + \alpha + n, b, \sigma ; \end{matrix} x \right] t^n \\ &= (1 + \xi)^\alpha {}_3F_2 \left[\begin{matrix} \frac{\alpha}{b+1}, 1 + \alpha + \beta, \nu ; \\ 1 + \frac{\alpha}{b+1}, \sigma ; \end{matrix} -x\xi \right] \end{aligned} \quad (4.4.9)$$

where ξ is given by equation (4.4.4).

Now using equation (2.1.20) in above equation, we get :

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\alpha}{\{\alpha + (b + 1)n\}} H_n^{(\alpha+bn, \beta-bn-n)}[\nu, \sigma, x] t^n \\ &= (1 + \xi)^\alpha {}_3F_2 \left[\begin{matrix} \frac{\alpha}{b+1}, 1 + \alpha + \beta, \nu ; \\ 1 + \frac{\alpha}{b+1}, \sigma ; \end{matrix} x(-\xi) \right] \end{aligned} \quad (4.4.10)$$

where ξ is given by equation (4.4.4).

4.5 New Applications of First Generating Relation(4.2.1)

(i). Putting $\beta = 0$ and $\zeta = \Theta = \frac{t}{1-t}$ from equation (2.1.24) in equation (4.2.1), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + n)} \binom{\alpha + n}{n} {}_{p+m}F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha), (b_q) & ; & \end{matrix} \right] t^n \\ &= (1-t)^{-\alpha} \left\{ \frac{\lambda}{\alpha} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m}, (a_p) & ; & x \left(\frac{-t}{1-t} \right)^m \\ 1 + \frac{\alpha}{m}, (b_q) & ; & \end{matrix} \right] \right. \\ & \quad \left. + \frac{\mu t}{(1-t)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m}, 1 + \frac{t\alpha}{m}, (a_p) & ; & x \left(\frac{-t}{1-t} \right)^m \\ 1 + \frac{\alpha}{m}, \frac{t\alpha}{m}, (b_q) & ; & \end{matrix} \right] \right\} \quad (4.5.1) \end{aligned}$$

(ii). Putting $\beta = 1$ and $\zeta = \Lambda = \frac{1-2t-\sqrt{(1-4t)}}{2t}$ from equation (2.1.25) in equation (4.2.1), we find

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + 2n)} \binom{\alpha + 2n}{n} {}_{p+m}F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha + n), (b_q) & ; & \end{matrix} \right] t^n \\ &= (1+\Lambda)^{\alpha} \left\{ \frac{\lambda}{\alpha} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{2m}, (a_p) & ; & x (-\Lambda)^m \\ 1 + \frac{\alpha}{2m}, (b_q) & ; & \end{matrix} \right] \right. \\ & \quad \left. + \frac{\mu \Lambda}{(1-\Lambda)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{2m}, 1 + \frac{\alpha \Lambda}{m(\Lambda+1)}, (a_p) & ; & x (-\Lambda)^m \\ 1 + \frac{\alpha}{2m}, \frac{\alpha \Lambda}{m(\Lambda+1)}, (b_q) & ; & \end{matrix} \right] \right\} \quad (4.5.2) \end{aligned}$$

(iii). Putting $\beta = -2$ and $\zeta = \Xi = \frac{-1+\sqrt{(1+4t)}}{2}$ from equation (2.1.26) in equation (4.2.1), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - n)} \binom{\alpha - n}{n} {}_{p+m}F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - 2n), (b_q) & ; & \end{matrix} \right] t^n \\ &= (1+\Xi)^{\alpha} \left\{ \frac{\lambda}{\alpha} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{-\alpha}{m}, (a_p) & ; & x (-\Xi)^m \\ 1 - \frac{\alpha}{m}, (b_q) & ; & \end{matrix} \right] \right. \\ & \quad \left. + \frac{\mu \Xi}{(1+2\Xi)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{-\alpha}{m}, 1 + \frac{\alpha \Xi}{m(\Xi+1)}, (a_p) & ; & x (-\Xi)^m \\ 1 - \frac{\alpha}{m}, \frac{\alpha \Xi}{m(\Xi+1)}, (b_q) & ; & \end{matrix} \right] \right\} \quad (4.5.3) \end{aligned}$$

(iv). Putting $\beta = -3$ and $\zeta = \Upsilon$ from equation (2.1.27) in equation (4.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - 2n)} \binom{\alpha - 2n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; \\ \Delta(m; 1 + \alpha - 3n), (b_q); & x \end{matrix} \right] t^n \\ &= (1 + \Upsilon)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{matrix} \frac{-\alpha}{2m}, (a_p); & x(-\Upsilon)^m \\ 1 - \frac{\alpha}{2m}, (b_q); & \end{matrix} \right] \right. \\ & \quad \left. + \frac{\mu \Upsilon}{(1 + 3\Upsilon)^{p+2}} F_{q+2} \left[\begin{matrix} \frac{-\alpha}{2m}, 1 + \frac{\alpha \Upsilon}{m(\Upsilon+1)}, (a_p); & x(-\Upsilon)^m \\ 1 - \frac{\alpha}{2m}, \frac{\alpha \Upsilon}{m(\Upsilon+1)}, (b_q); & \end{matrix} \right] \right\} \end{aligned} \quad (4.5.4)$$

(v). Putting $\beta = \frac{-1}{2}$ and $\zeta = U$ from equation (2.1.28) in equation (4.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{2}n)} \binom{\alpha + \frac{1}{2}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; \\ \Delta(m; 1 + \alpha - \frac{1}{2}n), (b_q); & x \end{matrix} \right] t^n \\ &= (1 + U)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{matrix} \frac{2\alpha}{m}, (a_p); & x(-U)^m \\ 1 + \frac{2\alpha}{m}, (b_q); & \end{matrix} \right] \right. \\ & \quad \left. + \frac{\mu U}{(1 + \frac{1}{2}U)^{p+2}} F_{q+2} \left[\begin{matrix} \frac{2\alpha}{m}, 1 + \frac{\alpha U}{m(U+1)}, (a_p); & x(-U)^m \\ 1 + \frac{2\alpha}{m}, \frac{\alpha U}{m(U+1)}, (b_q); & \end{matrix} \right] \right\} \end{aligned} \quad (4.5.5)$$

(vi). Putting $\beta = \frac{-3}{2}$ and $\zeta = \Psi$ from equation (2.1.29) in equation (4.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - \frac{1}{2}n)} \binom{\alpha - \frac{1}{2}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; \\ \Delta(m; 1 + \alpha - \frac{3}{2}n), (b_q); & x \end{matrix} \right] t^n \\ &= (1 + \Psi)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{matrix} \frac{-2\alpha}{m}, (a_p); & x(-\Psi)^m \\ 1 - \frac{2\alpha}{m}, (b_q); & \end{matrix} \right] \right. \\ & \quad \left. + \frac{\mu \Psi}{(1 + \frac{3}{2}\Psi)^{p+2}} F_{q+2} \left[\begin{matrix} \frac{-2\alpha}{m}, 1 + \frac{\alpha \Psi}{m(\Psi+1)}, (a_p); & x(-\Psi)^m \\ 1 - \frac{2\alpha}{m}, \frac{\alpha \Psi}{m(\Psi+1)}, (b_q); & \end{matrix} \right] \right\} \end{aligned} \quad (4.5.6)$$

(vii). Putting $\beta = \frac{-1}{3}$ and $\zeta = \Pi$ from equation (2.1.30) in equation (4.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{2}{3}n)} \binom{\alpha + \frac{2}{3}n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; \\ \Delta(m; 1 + \alpha - \frac{1}{3}n), (b_q); & x \end{matrix} \right] t^n \\ &= (1 + \Pi)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{matrix} \frac{3\alpha}{2m}, (a_p); & x(-\Pi)^m \\ 1 + \frac{3\alpha}{2m}, (b_q); & \end{matrix} \right] \right. \end{aligned}$$

$$+ \frac{\mu \Pi}{(1 + \frac{1}{3} \Pi)} {}^{p+2}F_{q+2} \left[\begin{matrix} \frac{3\alpha}{2m}, 1 + \frac{\alpha \Pi}{m(\Pi+1)}, (a_p); \\ 1 + \frac{3\alpha}{2m}, \frac{\alpha \Pi}{m(\Pi+1)}, (b_q); \end{matrix} x(-\Pi)^m \right] \Bigg\} \quad (4.5.7)$$

(viii). Putting $\beta = \frac{-2}{3}$ and $\zeta = \Phi$ from equation (2.1.31) in equation (4.2.1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{3}n)} \binom{\alpha + \frac{1}{3}n}{n} {}^{p+m}F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) & ; & x \\ \Delta(m; 1 + \alpha - \frac{2}{3}n), (b_q); \end{matrix} \right] t^n \\ &= (1 + \Phi)^\alpha \left\{ \frac{\lambda}{\alpha} {}^{p+1}F_{q+1} \left[\begin{matrix} \frac{3\alpha}{m}, (a_p); \\ 1 + \frac{3\alpha}{m}, (b_q); \end{matrix} x(-\Phi)^m \right] \right. \\ & \quad \left. + \frac{\mu \Phi}{(1 + \frac{2}{3} \Phi)} {}^{p+2}F_{q+2} \left[\begin{matrix} \frac{3\alpha}{m}, 1 + \frac{\alpha \Phi}{m(\Phi+1)}, (a_p); \\ 1 + \frac{3\alpha}{m}, \frac{\alpha \Phi}{m(\Phi+1)}, (b_q); \end{matrix} x(-\Phi)^m \right] \right\} \quad (4.5.8) \end{aligned}$$

4.6 New Applications of Second Generating Relation (4.2.7)

The following Generating relations of this section are believed to be new in author's knowledge and are not available in the literature of Generating relations.

(i). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}$ in equation (4.2.7), using the definition of $\mathcal{B}_n^{(\alpha, \beta)}(x; m, 1, \frac{\beta+1}{\alpha})$ and after simplifying, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)n}{n} {}^{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\alpha + (\beta + 1)n}{(\beta + 1)(\beta + 1 - m)}, (a_p); \\ \Delta(m; 1 + \alpha + \beta n), \frac{\alpha + (\beta + 1)n}{(\beta + 1)(\beta + 1 - m)}, (b_q); \end{matrix} x \right] t^n \\ &= (1 + \zeta)^\alpha \left\{ {}^{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m(\beta + 1)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta + 1)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\ & \quad \left. + \frac{(\beta + 1)\zeta}{(1 - \beta\zeta)} {}^{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta + 1)}, 1 + \frac{\alpha\zeta}{(\beta + 1)(1 + m\zeta - \beta\zeta)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta + 1)}, \frac{\alpha\zeta}{(\beta + 1)(1 + m\zeta - \beta\zeta)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (4.6.1) \end{aligned}$$

where ζ is given by equations (2.1.3) and (2.1.4).

(ii). Putting $\lambda = 1, \mu = \frac{\beta+1}{\alpha}, m = 1$ in equation (4.2.7) and using the definition of $\mathcal{B}_n^{(\alpha, \beta)}(x; 1, 1, \frac{\beta+1}{\alpha})$, we get

$$\sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)n}{n} {}^{p+2}F_{q+2} \left[\begin{matrix} -n, 1 + \frac{\alpha + (\beta + 1)n}{(\beta + 1)\beta}, (a_p); \\ 1 + \alpha + \beta n, \frac{\alpha + (\beta + 1)n}{(\beta + 1)\beta}, (b_q); \end{matrix} x \right] t^n$$

$$\begin{aligned}
&= (1 + \zeta)^\alpha \left\{ {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{(\beta+1)}, (a_p); \\ 1 + \frac{\alpha}{(\beta+1)}, (b_q); \end{matrix} -x\zeta \right] \right. \\
&\quad \left. + \frac{(\beta+1)\zeta}{(1-\beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{(\beta+1)}, 1 + \frac{\alpha\zeta}{(\beta+1)(1+\zeta-\beta\zeta)}, (a_p); \\ 1 + \frac{\alpha}{(\beta+1)}, \frac{\alpha\zeta}{(\beta+1)(1+\zeta-\beta\zeta)}, (b_q); \end{matrix} -x\zeta \right] \right\} \quad (4.6.2)
\end{aligned}$$

where ζ is given by equations (2.1.3) and (2.1.4).

(iii). Putting $\lambda = 1, \mu = \frac{1}{2\alpha}, \beta = \frac{-1}{2}, m = 1$ in equation (4.2.7), using the definition of $\mathcal{B}_n^{(\alpha, -\frac{1}{2})}(x; 1, 1, \frac{1}{2\alpha})$ and replacing ζ by U from equation (2.1.28), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \binom{\alpha + \frac{n}{2}}{n} {}_{p+2}F_{q+2} \left[\begin{matrix} -n, 1 - 4\alpha - 2n, (a_p); \\ 1 + \alpha - \frac{n}{2}, -4\alpha - 2n, (b_q); \end{matrix} x \right] t^n \\
&= (1 + U)^\alpha \left\{ {}_{p+1}F_{q+1} \left[\begin{matrix} 2\alpha, (a_p); \\ 1 + 2\alpha, (b_q); \end{matrix} -xU \right] \right. \\
&\quad \left. + \frac{U}{(2+U)} {}_{p+2}F_{q+2} \left[\begin{matrix} 2\alpha, 1 + \frac{4\alpha U}{(2+3U)}, (a_p); \\ 1 + 2\alpha, \frac{4\alpha U}{(2+3U)}, (b_q); \end{matrix} -xU \right] \right\} \quad (4.6.3)
\end{aligned}$$

(iv). Putting $\beta = 0$ and $\zeta = \Theta = \frac{t}{1-t}$ from equation (2.1.24) in equation (4.2.7), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + n)} \binom{\alpha + n}{n} {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(1-m)}, (a_p); \\ \Delta(m; 1 + \alpha), \frac{\lambda + \mu n}{\mu(1-m)}, (b_q); \end{matrix} x \right] t^n \\
&= (1 - t)^{-\alpha} \left\{ \frac{\lambda}{\alpha} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m}, (a_p); \\ 1 + \frac{\alpha}{m}, (b_q); \end{matrix} x \left(\frac{-t}{1-t} \right)^m \right] + \right. \\
&\quad \left. \frac{\mu t}{(1-t)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m}, 1 + \frac{t\alpha}{\{1+t(m-1)\}}, (a_p); \\ 1 + \frac{\alpha}{m}, \frac{t\alpha}{\{1+t(m-1)\}}, (b_q); \end{matrix} x \left(\frac{-t}{1-t} \right)^m \right] \right\} \quad (4.6.4)
\end{aligned}$$

(v). Putting $\beta = 1$ and $\zeta = \Lambda = \frac{1-2t-\sqrt{(1-4t)}}{2t}$ from equation (2.1.25) in equation (4.2.7), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + 2n)} \binom{\alpha + 2n}{n} {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(2-m)}, (a_p); \\ \Delta(m; 1 + \alpha + n), \frac{\lambda + \mu n}{\mu(2-m)}, (b_q); \end{matrix} x \right] t^n \\
&= (1 + \Lambda)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{2m}, (a_p); \\ 1 + \frac{\alpha}{2m}, (b_q); \end{matrix} x(-\Lambda)^m \right] \right\}
\end{aligned}$$

$$+ \frac{\mu\Lambda}{(1-\Lambda)^{p+2}} F_{q+2} \left[\begin{array}{c} \frac{\alpha}{2m}, 1 + \frac{\Lambda\alpha}{2(1+m\Lambda-\Lambda)}, (a_p); \\ 1 + \frac{\alpha}{2m}, \frac{\Lambda\alpha}{2(1+m\Lambda-\Lambda)}, (b_q); \end{array} x(-\Lambda)^m \right] \quad (4.6.5)$$

(vi). Putting $\beta = -2$ and $\zeta = \Xi = \frac{-1+\sqrt{(1+4t)}}{2}$ from equation (2.1.26) in equation (4.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - n)} \binom{\alpha - n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 - \frac{\lambda + \mu n}{\mu(1+m)}, (a_p) \ ; \ x \\ \Delta(m; 1 + \alpha - 2n), \frac{-(\lambda + \mu n)}{\mu(1+m)}, (b_q); \end{array} \right] t^n \\ &= (1 + \Xi)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{array}{c} \frac{-\alpha}{m}, (a_p) \ ; \ x(-\Xi)^m \\ 1 - \frac{\alpha}{m}, (b_q); \end{array} \right] \right. \\ & \quad \left. + \frac{\mu \Xi}{(1 + 2\Xi)^{p+2}} F_{q+2} \left[\begin{array}{c} \frac{-\alpha}{m}, 1 - \frac{\Xi\alpha}{(1+m\Xi+2\Xi)}, (a_p); \ x(-\Xi)^m \\ 1 - \frac{\alpha}{m}, \frac{-\Xi\alpha}{(1+m\Xi+2\Xi)}, (b_q); \end{array} \right] \right\} \quad (4.6.6) \end{aligned}$$

(vii). Putting $\beta = -3$ and $\zeta = \Upsilon$ from equation (2.1.27) in equation (4.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - 2n)} \binom{\alpha - 2n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 - \frac{(\lambda + \mu n)}{\mu(2+m)}, (a_p); \ x \\ \Delta(m; 1 + \alpha - 3n), -\frac{(\lambda + \mu n)}{\mu(2+m)}, (b_q); \end{array} \right] t^n \\ &= (1 + \Upsilon)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{array}{c} \frac{-\alpha}{2m}, (a_p) \ ; \ x(-\Upsilon)^m \\ 1 - \frac{\alpha}{2m}, (b_q); \end{array} \right] \right. \\ & \quad \left. + \frac{\mu \Upsilon}{(1 + 3\Upsilon)^{p+2}} F_{q+2} \left[\begin{array}{c} \frac{-\alpha}{2m}, 1 - \frac{\alpha\Upsilon}{2(1+m\Upsilon+3\Upsilon)}, (a_p); \ x(-\Upsilon)^m \\ 1 - \frac{\alpha}{2m}, \frac{-\alpha\Upsilon}{2(1+m\Upsilon+3\Upsilon)}, (b_q); \end{array} \right] \right\} \quad (4.6.7) \end{aligned}$$

(viii). Putting $\beta = \frac{-1}{2}$ and $\zeta = U$ from equation (2.1.28) in equation (4.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{2}n)} \binom{\alpha + \frac{1}{2}n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(\frac{1}{2}-m)}, (a_p); \ x \\ \Delta(m; 1 + \alpha - \frac{1}{2}n), \frac{\lambda + \mu n}{\mu(\frac{1}{2}-m)}, (b_q); \end{array} \right] t^n \\ &= (1 + U)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{array}{c} \frac{2\alpha}{m}, (a_p) \ ; \ x(-U)^m \\ 1 + \frac{2\alpha}{m}, (b_q); \end{array} \right] \right. \\ & \quad \left. + \frac{\mu U}{(1 + \frac{1}{2}U)^{p+2}} F_{q+2} \left[\begin{array}{c} \frac{2\alpha}{m}, 1 + \frac{2U\alpha}{(1+mU+\frac{1}{2}U)}, (a_p); \ x(-U)^m \\ 1 + \frac{2\alpha}{m}, \frac{2U\alpha}{(1+mU+\frac{1}{2}U)}, (b_q); \end{array} \right] \right\} \quad (4.6.8) \end{aligned}$$

(ix). Putting $\beta = \frac{-3}{2}$ and $\zeta = \Psi$ from equation (2.1.29) in equation (4.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha - \frac{1}{2}n)} \binom{\alpha - \frac{1}{2}n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 - \frac{(\lambda + \mu n)}{\mu(\frac{1}{2} + m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{3}{2}n), -\frac{(\lambda + \mu n)}{\mu(\frac{1}{2} + m)}, (b_q); \end{array} x \right] t^n \\ &= (1 + \Psi)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{array}{c} \frac{-2\alpha}{m}, (a_p); \\ 1 - \frac{2\alpha}{m}, (b_q); \end{array} x(-\Psi)^m \right] \right. \\ & \quad \left. + \frac{\mu \Psi}{(1 + \frac{3}{2}\Psi)^{p+2}} F_{q+2} \left[\begin{array}{c} \frac{-2\alpha}{m}, 1 - \frac{2\Psi\alpha}{(1+m\Psi + \frac{3}{2}\Psi)}, (a_p); \\ 1 - \frac{2\alpha}{m}, \frac{-2\Psi\alpha}{(1+m\Psi + \frac{3}{2}\Psi)}, (b_q); \end{array} x(-\Psi)^m \right] \right\} \quad (4.6.9) \end{aligned}$$

(x). Putting $\beta = \frac{-1}{3}$ and $\zeta = \Pi$ from equation (2.1.30) in equation (4.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{2}{3}n)} \binom{\alpha + \frac{2}{3}n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(\frac{2}{3} - m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{1}{3}n), \frac{\lambda + \mu n}{\mu(\frac{2}{3} - m)}, (b_q); \end{array} x \right] t^n \\ &= (1 + \Pi)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{array}{c} \frac{3\alpha}{2m}, (a_p); \\ 1 + \frac{3\alpha}{2m}, (b_q); \end{array} x(-\Pi)^m \right] \right. \\ & \quad \left. + \frac{\mu \Pi}{(1 + \frac{1}{3}\Pi)^{p+2}} F_{q+2} \left[\begin{array}{c} \frac{3\alpha}{2m}, 1 + \frac{3\Pi\alpha}{2(1+m\Pi + \frac{1}{3}\Pi)}, (a_p); \\ 1 + \frac{3\alpha}{2m}, \frac{3\Pi\alpha}{2(1+m\Pi + \frac{1}{3}\Pi)}, (b_q); \end{array} x(-\Pi)^m \right] \right\} \quad (4.6.10) \end{aligned}$$

(xi). Putting $\beta = \frac{-2}{3}$ and $\zeta = \Phi$ from equation (2.1.31) in equation (4.2.7), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{(\alpha + \frac{1}{3}n)} \binom{\alpha + \frac{1}{3}n}{n}_{p+m+1} F_{q+m+1} \left[\begin{array}{c} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(\frac{1}{3} - m)}, (a_p); \\ \Delta(m; 1 + \alpha - \frac{2}{3}n), \frac{\lambda + \mu n}{\mu(\frac{1}{3} - m)}, (b_q); \end{array} x \right] t^n \\ &= (1 + \Phi)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1}} F_{q+1} \left[\begin{array}{c} \frac{3\alpha}{m}, (a_p); \\ 1 + \frac{3\alpha}{m}, (b_q); \end{array} x(-\Phi)^m \right] \right. \\ & \quad \left. + \frac{\mu \Phi}{(1 + \frac{2}{3}\Phi)^{p+2}} F_{q+2} \left[\begin{array}{c} \frac{3\alpha}{m}, 1 + \frac{3\Phi\alpha}{(1+m\Phi + \frac{2}{3}\Phi)}, (a_p); \\ 1 + \frac{3\alpha}{m}, \frac{3\Phi\alpha}{(1+m\Phi + \frac{2}{3}\Phi)}, (b_q); \end{array} x(-\Phi)^m \right] \right\} \quad (4.6.11) \end{aligned}$$

Making suitable adjustments of parameters and variables in all generating relations of section 4.5 and 4.6, we can also obtain a number of new generating relations involving restricted generalized Laguerre polynomials, restricted Jacobi polynomials, restricted generalized Rice polynomials of Khandekar and other orthogonal polynomials.

Chapter 5

Some Transformation, Reduction and Summation Formulae

Associated With Hypergeometric Polynomials-I

5.1 Introduction

In this chapter, we have introduced five new transformations for Srivastava-Daoust double hypergeometric polynomials with $-1, +1$ arguments using Beta integral technique in the evaluation of the integral $\int_{-1}^{+1} (1+x)^b(1-x)^c P_n^{(\alpha, \beta)}(x) dx$ by different approaches. A summation theorem for hypergeometric polynomials ${}_5F_4(1)$ in terms of Pochhammer's symbols, some special cases of main integrals and six reduction formulae for hypergeometric polynomials ${}_5F_4(-1)$ in terms of hypergeometric polynomials ${}_3F_2(1)$, are also deduced.

The Jacobi Polynomials of first kind $P_n^{(\alpha, \beta)}(x)$ are defined by the following equations [85, p.254 (132.1, 132.2, 132.3), p.255 (132.7, 132.8, 132.9), p.274(142.5)]:

$$P_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1+\alpha+\beta+n; \\ 1+\alpha \end{matrix} ; \frac{1-x}{2} \right] \quad (5.1.1)$$

$$= \frac{(1+\alpha)_n}{n!} \left(\frac{x+1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -\beta-n; \\ 1+\alpha \end{matrix} ; \frac{x-1}{x+1} \right] \quad (5.1.2)$$

$$= \frac{(-1)^n(1+\beta)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1+\alpha+\beta+n; \\ 1+\beta \end{matrix} ; \frac{1+x}{2} \right] \quad (5.1.3)$$

$$= \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n} \left(\frac{x-1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -\alpha-n; \\ -\alpha-\beta-2n; \end{matrix} ; \frac{2}{1-x} \right] \quad (5.1.4)$$

$$= \frac{(1+\beta)_n}{n!} \left(\frac{x-1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -\alpha-n; \\ 1+\beta \end{matrix} ; \frac{x+1}{x-1} \right] \quad (5.1.5)$$

$$= \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n} \left(\frac{x+1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -\beta-n; \\ -\alpha-\beta-2n; \end{matrix} ; \frac{2}{x+1} \right] \quad (5.1.6)$$

$$= \frac{(1+\alpha)_n}{(1+\alpha+\beta)_n} \sum_{k=0}^n \frac{(-1)^{n-k}(\beta-\alpha)_{n-k}(1+\alpha+\beta)_{n+k}(1+2\alpha)_k(1+2\alpha+2k)}{(n-k)!(1+2\alpha)_{n+k+1}(1+\alpha)_k} \cdot P_k^{(\alpha, \alpha)}(x) \quad (5.1.7)$$

where n is a non-negative integer.

An integral reducible to Beta function [85, p.31, Q.no.7]

$$\int_{-1}^{+1} (1-x)^{p-1} (1+x)^{q-1} dx = 2^{p+q-1} B(p, q) = \frac{2^{p+q-1} \Gamma(p) \Gamma(q)}{\Gamma(p+q)} \quad (5.1.8)$$

where $\Re(p) > 0$, $\Re(q) > 0$.

Gauss summation theorem [85, p.49, Th.18]

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad (5.1.9)$$

($\Re(c-a-b) > 0$; $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$)

Saalschütz' summation theorem for terminating series [85, p.87, Th.29]

$${}_3F_2(-n, a, b; c, 1+a+b-c-n; 1) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} \quad (5.1.10)$$

($n \in \mathbb{N}_0$; $c, 1+a+b-c-n, c-a+n, c-b+n, c-a-b \in \mathbb{C} \setminus \mathbb{Z}_0^-$).

Summation Property [117, p.103(2.1.18)]

$$\sum_{k=0}^n \sum_{r=0}^k \phi(r, k) = \sum_{k=0}^n \sum_{r=0}^{n-k} \phi(r, k+r) \quad (5.1.11)$$

One definite integral [83, p.7(4.1)]

$$\int_{-1}^{+1} (1-x)^\alpha (1+x)^b P_n^{(\alpha, \beta)}(x) dx = \frac{2^{\alpha+b+1} \Gamma(b+1) \Gamma(\alpha+n+1) \Gamma(b-\beta+1)}{\Gamma(b-\beta-n+1) \Gamma(\alpha+b+n+2) n!} \quad (5.1.12)$$

where $\Re(\alpha) > -1$, $\Re(b) > -1$.

Srivastava-Daoust Function [116, p.37(1.4.21, 1.4.22, 1.4.23)]

A further generalization of the Kampé de Fériet function is due to Srivastava and Daoust who indeed defined an extension of Wright's ${}_p\Psi_q$ function in two variables. More generally, we recall here the following extension of the Wright function ${}_p\Psi_q$ in *several* variables, which is referred to in the literature as the generalized Lauricella function of several variables; it is due also to Srivastava and Daoust [113, p.454]:

$$F_{C:D'; \dots; D^{(n)}}^A : B' : \dots; B^{(n)} \left(\begin{array}{l} [(a) : \theta', \dots, \theta^{(n)}] : [(b') : \phi']; \dots; [(b^{(n)}) : \phi^{(n)}]; \\ \\ [(c) : \psi', \dots, \psi^{(n)}] : [(d') : \delta']; \dots; [(d^{(n)}) : \delta^{(n)}]; \end{array} \right. \quad z_1, \dots, z_n$$

$$= \sum_{m_1, \dots, m_n=0}^{\infty} \Omega(m_1, \dots, m_n) \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_n^{m_n}}{m_n!} \quad (5.1.13)$$

where, for convenience,

$$\Omega(m_1, \dots, m_n) = \frac{\prod_{j=1}^A (a_j)_{m_1 \theta'_j + \dots + m_n \theta_j^{(n)}} \prod_{j=1}^{B'} (b'_j)_{m_1 \phi'_j} \cdots \prod_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_n \phi_j^{(n)}}}{\prod_{j=1}^C (c_j)_{m_1 \psi'_j + \dots + m_n \psi_j^{(n)}} \prod_{j=1}^{D'} (d'_j)_{m_1 \delta'_j} \cdots \prod_{j=1}^{D^{(n)}} (d_j^{(n)})_{m_n \delta_j^{(n)}}} \quad (5.1.14)$$

the coefficients

$$\begin{cases} \theta_j^{(k)}, j = 1, \dots, A; & \phi_j^{(k)}, j = 1, \dots, B^{(k)}; & \psi_j^{(k)}, j = 1, \dots, C; \\ \delta_j^{(k)}, j = 1, \dots, D^{(k)}; & \forall k \in \{1, \dots, n\} \end{cases} \quad (5.1.15)$$

are real and positive, and (a) abbreviates the array of A parameters $a_1, \dots, a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}, j = 1, \dots, B^{(k)}; \forall k \in \{1, \dots, n\}$$

with similar interpretations for (c) and $(d^{(k)})$, $k = 1, \dots, n$; *et cetera*. For the precise conditions under which the multiple hypergeometric series in the definition (5.1.13) converges absolutely, see [37, Section 3.7 ; 38, Section 1.4; 114, pp.153 – 157, Sections 3, 4, 5].

Clausen's hypergeometric series

Putting $p = 3, q = 2$ in equation (1.3.3), we get Clausen's hypergeometric series.

Kummer's confluent hypergeometric series

Putting $p = 1, q = 1$ in equation (1.3.3), we get Kummer's confluent hypergeometric series.

Gauss's hypergeometric series

Putting $p = 2, q = 1$ in equation (1.3.3), we get Gauss's hypergeometric series.

Watson's theorem for terminating and non-terminating series [117, p.95, Q.26; see also 2, p.16]

$${}_3F_2 \left[\begin{matrix} a, b, c; \\ \frac{a+b+1}{2}, 2c; \end{matrix} \quad 1 \right] = \frac{\Gamma(\frac{1}{2})\Gamma(c + \frac{1}{2})\Gamma(\frac{a+b+1}{2})\Gamma(c - \frac{a}{2} - \frac{b}{2} + \frac{1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})\Gamma(c - \frac{a}{2} + \frac{1}{2})\Gamma(c - \frac{b}{2} + \frac{1}{2})} \quad (5.1.16)$$

$$(\Re(c - \frac{a}{2} - \frac{b}{2}) > -\frac{1}{2}; \frac{1+a+b}{2}, 2c \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

Dixon's theorem for terminating and non-terminating series [85, p.92, Th.33]

$${}_3F_2 \left[\begin{matrix} a, & b, & c & ; & 1 \\ 1+a-b, & 1+a-c; \end{matrix} \right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2})\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a)\Gamma(1+a-b-c)} \quad (5.1.17)$$

$$(\Re(a-2b-2c) > -2; 1+a-b, 1+a-c \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

Whipple's theorem for non-terminating series[2, p.16]

$${}_3F_2 \left[\begin{matrix} a, & b, & c; \\ g, & f; \end{matrix} 1 \right] = \frac{\pi \Gamma(g)\Gamma(f)}{2^{2c-1}\Gamma(\frac{a+g}{2})\Gamma(\frac{a+f}{2})\Gamma(\frac{b+g}{2})\Gamma(\frac{b+f}{2})} \quad (5.1.18)$$

$$(\Re(c) > 0; a+b=1, g+f=2c+1; g, f \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

Whipple's theorem for terminating series

$${}_3F_2 \left[\begin{matrix} -m, & a, & 1-a; \\ c, & 1-c-2m; \end{matrix} 1 \right] = \frac{\left(\frac{a+c}{2}\right)_m \left(\frac{c-a+1}{2}\right)_m}{\left(\frac{c}{2}\right)_m \left(\frac{c+1}{2}\right)_m} \quad (5.1.19)$$

$$(m \in \mathbb{N}_0; c, 1-c-2m, \frac{a+c+2m}{2}, \frac{1-a+c+2m}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

5.2 Transformations For Srivastava-Daoust Double Hypergeometric Polynomials

If n is non-negative integer and any values of parameters leading to the results which do not make sense, are tacitly excluded, then

$$F_{3:0;2}^{4:0;1} \left(\begin{matrix} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 1, 2] : \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1] : \\ \text{---}; [c+1 : 1] ; \\ \text{---}; [1+\alpha : 1], [c+b+2 : 1]; \end{matrix} \begin{matrix} -1, 1 \end{matrix} \right)$$

$$\begin{aligned}
&= F_{4: 1; 1}^{5: 1; 1} \left(\begin{array}{l} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 1, 1], \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1], \\ [\alpha + 1 : 1, 1] \quad : [b + 1 : 1]; [c + 1 : 1]; \\ -1, 1 \\ [c + b + 2 : 1, 1] : [\alpha + 1 : 1]; [\alpha + 1 : 1]; \end{array} \right) \quad (5.2.1)
\end{aligned}$$

$$\begin{aligned}
&= F_{3: 0; 2}^{4: 0; 1} \left(\begin{array}{l} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 1, 2] : \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1] \quad : \\ \text{---}; \quad [b + 1 : 1] \quad ; \\ 1, -1 \\ \text{---}; [1 + \alpha : 1], [c + b + 2 : 1]; \end{array} \right) \quad (5.2.2)
\end{aligned}$$

$$\begin{aligned}
&= F_{3: 2; 0}^{4: 1; 0} \left(\begin{array}{l} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 2, 1] : \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1] \quad : \\ [c + 1 : 1] \quad ; \text{---}; \\ 1, -1 \\ [1 + \alpha : 1], [c + b + 2 : 1]; \text{---}; \end{array} \right) \quad (5.2.3)
\end{aligned}$$

$$\begin{aligned}
&= F_{4: 1; 1}^{5: 1; 1} \left(\begin{array}{l} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 1, 1], \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1], \\ [\alpha + 1 : 1, 1] \quad : [c + 1 : 1]; [b + 1 : 1] ; \\ 1, -1 \\ [c + b + 2 : 1, 1] : [\alpha + 1 : 1]; [\alpha + 1 : 1]; \end{array} \right) \quad (5.2.4)
\end{aligned}$$

$$\begin{aligned}
&= F_{3:2;0}^{4:1;0} \left(\begin{array}{l} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 2, 1] : \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1] : \\ [b + 1 : 1] \quad ; \text{---}; \\ \quad \quad \quad -1, 1 \\ [1 + \alpha : 1], [c + b + 2 : 1] ; \text{---}; \end{array} \right) \quad (5.2.5)
\end{aligned}$$

Proof:- Comparing the integral (5.4.2) with the five integrals (5.4.4), (5.4.6), (5.4.8), (5.4.10) and (5.4.12), we can obtain the above five hypergeometric transformations associated with Srivastava-Daoust double hypergeometric polynomials.

5.3 Summation Theorem For Hypergeometric Polynomials ${}_5F_4(1)$

Comparing equation (5.1.12) with the equation (5.5.2) and using some algebraic properties of Pochhammer symbols, after simplification, we get the following summation theorem:

$${}_5F_4 \left[\begin{array}{l} -n, \alpha - b, 2\alpha + 1, \alpha + \frac{3}{2}, \alpha + \beta + n + 1 \\ 2\alpha + n + 2, \alpha + b + 2, \alpha + \frac{1}{2}, \alpha - \beta - n + 1 \end{array} ; 1 \right] = \frac{(2\alpha + 2)_n (\beta - b)_n}{(\alpha + b + 2)_n (\beta - \alpha)_n} \quad (5.3.1)$$

where n is non-negative integer and remaining numerator and denominator parameters are neither zero nor negative integers.

5.4 Evaluation of the integral $\int_{-1}^{+1} (1+x)^b(1-x)^c P_n^{(\alpha, \beta)}(x) dx$ using the definition of $P_k^{(\alpha, \alpha)}(x)$

In each of the following results, any exceptional values of the parameters and variables, which would make the results invalid, are tacitly excluded.

$$\begin{aligned} & \{\Delta_n(b, c, \alpha, \beta)\}^{-1} \int_{-1}^{+1} (1+x)^b(1-x)^c P_n^{(\alpha, \beta)}(x) dx \\ &= \sum_{k=0}^n \frac{(-1)^k (-n)_k (2\alpha+1)_k (\alpha + \frac{3}{2})_k (\alpha + \beta + n + 1)_k}{k! (2\alpha + n + 2)_k (\alpha + \frac{1}{2})_k (\alpha - \beta - n + 1)_k} {}_3F_2 \left[\begin{matrix} -k, 2\alpha + k + 1, c + 1; \\ 1 + \alpha, c + b + 2; \end{matrix} \quad 1 \right] \end{aligned} \quad (5.4.1)$$

$$\begin{aligned} &= F_{3: 0; 2}^{4: 0; 1} \left(\begin{matrix} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 1, 2] : \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1] \end{matrix} : \right. \\ &\quad \left. \begin{matrix} \text{---}; & [c + 1 : 1] & ; \\ & & -1, 1 \end{matrix} \right) \end{aligned} \quad (5.4.2)$$

$$\begin{aligned} &= \sum_{k=0}^n \frac{(-1)^k (-n)_k (2\alpha+1)_k (\alpha + \beta + n + 1)_k}{k! (2\alpha + n + 2)_k (\alpha - \beta - n + 1)_k} \frac{(\alpha + \frac{3}{2})_k (b + 1)_k}{(\alpha + \frac{1}{2})_k (b + c + 2)_k} \times \\ &\quad \times {}_3F_2 \left[\begin{matrix} -k, -\alpha - k, c + 1; \\ 1 + \alpha, -b - k; \end{matrix} \quad 1 \right] \end{aligned} \quad (5.4.3)$$

$$\begin{aligned} &= F_{4: 1; 1}^{5: 1; 1} \left(\begin{matrix} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 1, 1], \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1], \\ [\alpha + 1 : 1, 1] : [b + 1 : 1]; [c + 1 : 1] ; \\ [c + b + 2 : 1, 1] : [\alpha + 1 : 1]; [\alpha + 1 : 1]; \end{matrix} \right. \\ &\quad \left. -1, 1 \right) \end{aligned} \quad (5.4.4)$$

$$= \sum_{k=0}^n \frac{(-n)_k (2\alpha+1)_k (\alpha+\beta+n+1)_k (\alpha+\frac{3}{2})_k}{k! (2\alpha+n+2)_k (\alpha-\beta-n+1)_k (\alpha+\frac{1}{2})_k} {}_3F_2 \left[\begin{matrix} -k, 2\alpha+k+1, b+1; \\ 1+\alpha, c+b+2; \end{matrix} \right]_1 \quad (5.4.5)$$

$$= F_{3: 0; 2}^{4: 0; 1} \left(\begin{matrix} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 1, 2] : \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1] : \\ \text{---}; [b + 1 : 1] ; \\ 1, -1 \\ \text{---}; [1 + \alpha : 1], [c + b + 2 : 1]; \end{matrix} \right) \quad (5.4.6)$$

$$= \sum_{k=0}^n \frac{(-n)_k (\alpha + \beta + n + 1)_k}{k! (2\alpha + n + 2)_k (\alpha - \beta - n + 1)_k} \frac{(\alpha + \frac{3}{2})_k (c + 1)_k 4^k}{(b + c + 2)_k} \times \\ \times {}_3F_2 \left[\begin{matrix} -k, -\alpha - k, -b - c - k - 1; \\ -2\alpha - 2k, -c - k; \end{matrix} \right]_1 \quad (5.4.7)$$

$$= F_{3: 2; 0}^{4: 1; 0} \left(\begin{matrix} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 2, 1] : \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1] : \\ [c + 1 : 1] ; \text{---}; \\ 1, -1 \\ [1 + \alpha : 1], [c + b + 2 : 1]; \text{---}; \end{matrix} \right) \quad (5.4.8)$$

$$= \sum_{k=0}^n \frac{(-n)_k (2\alpha+1)_k (\alpha+\beta+n+1)_k (\alpha+\frac{3}{2})_k (c+1)_k}{k! (2\alpha+n+2)_k (\alpha-\beta-n+1)_k (\alpha+\frac{1}{2})_k (b+c+2)_k} \times \\ \times {}_3F_2 \left[\begin{matrix} -k, -\alpha - k, b + 1; \\ 1 + \alpha, -c - k; \end{matrix} \right]_1 \quad (5.4.9)$$

$$= F_{4: 1; 1}^{5: 1; 1} \left(\begin{matrix} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 1, 1], \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1], \end{matrix} \right)$$

$$\left. \begin{array}{l} [\alpha + 1 : 1, 1] : [c + 1 : 1]; [b + 1 : 1] ; \\ [c + b + 2 : 1, 1] : [\alpha + 1 : 1]; [\alpha + 1 : 1]; \end{array} \right\} \begin{array}{l} 1, -1 \\ -1, 1 \end{array} \quad (5.4.10)$$

$$= \sum_{k=0}^n \frac{(-1)^k (-n)_k (\alpha + \beta + n + 1)_k (\alpha + \frac{3}{2})_k (b + 1)_k 4^k}{k! (2\alpha + n + 2)_k (\alpha - \beta - n + 1)_k (b + c + 2)_k} \times \\ \times {}_3F_2 \left[\begin{array}{c} -k, -\alpha - k, -b - c - k - 1; \\ -2\alpha - 2k, \quad -b - k; \end{array} \begin{array}{c} 1 \\ 1 \end{array} \right] \quad (5.4.11)$$

$$= F_{3: 2; 0}^{4: 1; 0} \left(\begin{array}{l} [-n : 1, 1], [\alpha + \frac{3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [2\alpha + 1 : 2, 1] : \\ [\alpha + \frac{1}{2} : 1, 1], [\alpha - \beta - n + 1 : 1, 1], [2\alpha + n + 2 : 1, 1] \end{array} ; \right. \\ \left. \begin{array}{l} [b + 1 : 1] ; \text{---}; \\ [1 + \alpha : 1], [c + b + 2 : 1] ; \text{---}; \end{array} \right\} \begin{array}{l} -1, 1 \\ -1, 1 \end{array} \quad (5.4.12)$$

where

$$\Delta_n(b, c, \alpha, \beta) = \frac{(-1)^n 2^{c+b+1} \Gamma(b+1) \Gamma(c+1) (1+\alpha)_n (\beta-\alpha)_n}{n! \Gamma(c+b+2) (2\alpha+2)_n}.$$

$$(\Re(b) > -1, \Re(c) > -1).$$

Proof:- Applying first, the definition (5.1.7) for $P_n^{(\alpha, \beta)}(x)$ and then definition (5.1.1) for Jacobi polynomials $P_k^{(\alpha, \alpha)}(x)$ in the integral $\int_{-1}^{+1} (1+x)^b (1-x)^c P_n^{(\alpha, \beta)}(x) dx$, expanding hypergeometric polynomial ${}_2F_1$ in series form, interchanging the order of integration and summation, using the Beta integral (5.1.8), we get the result (5.4.1) and applying the summation property (5.1.11) in the result (5.4.1) then writing the resulting double series in Srivastava-Daoust notation, we get the result (5.4.2).

Similarly, after using the definition (5.1.7) in $P_n^{(\alpha, \beta)}(x)$ in the integral $\int_{-1}^{+1} (1+x)^b (1-x)^c P_n^{(\alpha, \beta)}(x) dx$ and then using the remaining five definitions (5.1.2), (5.1.3), (5.1.4), (5.1.5) and (5.1.6) of Jacobi polynomials in $P_k^{(\alpha, \alpha)}(x)$ in place of the definition (5.1.1), we can evaluate the same integral in the forms of (5.4.3), (5.4.4); (5.4.5), (5.4.6); (5.4.7), (5.4.8); (5.4.9), (5.4.10) and (5.4.11), (5.4.12) respectively.

5.5 Special Cases of Main Integrals

Case (i): Putting $b = \alpha$ in equations (5.4.1) and (5.4.7), applying Saalschütz' summation theorem (5.1.10) and putting $b = \alpha$ in equations (5.4.3), (5.4.5), (5.4.9) and (5.4.11), applying Gauss theorem (5.1.9), we get

$$\begin{aligned} \int_{-1}^{+1} (1+x)^\alpha (1-x)^c P_n^{(\alpha, \beta)}(x) dx &= \frac{(-1)^n 2^{\alpha+c+1} \Gamma(c+1) \Gamma(\alpha+n+1) (\beta-\alpha)_n}{\Gamma(\alpha+c+2) (2\alpha+2)_n n!} \times \\ &\times {}_5F_4 \left[\begin{matrix} -n, \alpha-c, 2\alpha+1, \alpha+\frac{3}{2}, \alpha+\beta+n+1 \\ \alpha+c+2, 2\alpha+n+2, \alpha+\frac{1}{2}, \alpha-\beta-n+1 \end{matrix} ; -1 \right] \\ &(\Re(\alpha) > -1, \Re(c) > -1). \end{aligned} \quad (5.5.1)$$

which is believed to be new and is not found in the table [26, pp.284 – 288].

Case (ii): Putting $c = \alpha$ in equations (5.4.1), (5.4.3), (5.4.7) and (5.4.9), applying Gauss theorem (5.1.9) and putting $c = \alpha$ in equations (5.4.5) and (5.4.11), applying Saalschütz' summation theorem (5.1.10), we get

$$\begin{aligned} \int_{-1}^{+1} (1+x)^b (1-x)^\alpha P_n^{(\alpha, \beta)}(x) dx &= \frac{(-1)^n 2^{\alpha+b+1} \Gamma(b+1) \Gamma(\alpha+n+1) (\beta-\alpha)_n}{\Gamma(\alpha+b+2) (2\alpha+2)_n n!} \times \\ &\times {}_5F_4 \left[\begin{matrix} -n, \alpha-b, 2\alpha+1, \alpha+\frac{3}{2}, \alpha+\beta+n+1 \\ \alpha+b+2, 2\alpha+n+2, \alpha+\frac{1}{2}, \alpha-\beta-n+1 \end{matrix} ; 1 \right] \\ &(\Re(\alpha) > -1, \Re(b) > -1). \end{aligned} \quad (5.5.2)$$

5.6 Reduction Formulae For Hypergeometric Polynomials ${}_5F_4(-1)$

If n is non-negative integer and remaining numerator, denominator parameters are neither zero nor negative integers, then

$$\begin{aligned} &{}_5F_4 \left[\begin{matrix} -n, \alpha-c, 2\alpha+1, \alpha+\frac{3}{2}, \alpha+\beta+n+1 \\ \alpha+c+2, 2\alpha+n+2, \alpha+\frac{1}{2}, \alpha-\beta-n+1 \end{matrix} ; -1 \right] \\ &= \frac{(-1)^n (2\alpha+2)_n}{(\beta-\alpha)_n} {}_3F_2 \left[\begin{matrix} -n, \alpha+\beta+n+1, c+1 \\ \alpha+1, \alpha+c+2 \end{matrix} ; 1 \right] \end{aligned} \quad (5.6.1)$$

$$= \frac{(-1)^n(\alpha+1)_n(2\alpha+2)_n}{(\alpha+c+2)_n(\beta-\alpha)_n} {}_3F_2 \left[\begin{matrix} -n, -\beta-n, c+1; \\ \alpha+1, -\alpha-n \end{matrix}; 1 \right] \quad (5.6.2)$$

$$= \frac{(2\alpha+2)_n(1+\beta)_n}{(1+\alpha)_n(\beta-\alpha)_n} {}_3F_2 \left[\begin{matrix} -n, \alpha+\beta+n+1, \alpha+1; \\ \beta+1, \alpha+c+2 \end{matrix}; 1 \right] \quad (5.6.3)$$

$$= \frac{(2\alpha+2)_n(c+1)_n(1+\alpha+\beta)_{2n}}{(\beta-\alpha)_n(1+\alpha+\beta)_n(\alpha+c+2)_n(1+\alpha)_n} {}_3F_2 \left[\begin{matrix} -n, -\alpha-n, -c-\alpha-n-1; \\ -c-n, -\alpha-\beta-2n \end{matrix}; 1 \right] \quad (5.6.4)$$

$$= \frac{(c+1)_n(2\alpha+2)_n(1+\beta)_n}{(\alpha+1)_n(c+\alpha+2)_n(\beta-\alpha)_n} {}_3F_2 \left[\begin{matrix} -n, -\alpha-n, \alpha+1; \\ 1+\beta, -c-n \end{matrix}; 1 \right] \quad (5.6.5)$$

$$= \frac{(-1)^n(2\alpha+2)_n(1+\alpha+\beta)_{2n}}{(\beta-\alpha)_n(1+\alpha+\beta)_n(\alpha+c+2)_n} {}_3F_2 \left[\begin{matrix} -n, -\beta-n, -c-\alpha-n-1; \\ -\alpha-n, -\alpha-\beta-2n \end{matrix}; 1 \right] \quad (5.6.6)$$

Proof:- If we solve the integral in the left hand side of equation (5.5.1) using the definitions (5.1.1), (5.1.2), (5.1.3), (5.1.4), (5.1.5) and (5.1.6) of $P_n^{(\alpha, \beta)}(x)$ with the help of the Beta integral (5.1.8), then we get six reduction formulae for hypergeometric polynomials ${}_5F_4(-1)$ in terms of hypergeometric polynomials ${}_3F_2(1)$.

Remark:- It should be noted that we cannot find the sum of above hypergeometric polynomials ${}_3F_2$ in terms of gamma functions, using Saalschütz' theorem (5.1.10), Watson's theorem (5.1.16), Dixon's theorem (5.1.17), Whipple's theorem (5.1.19) and other summation theorems available in the literature of Special functions.

Chapter 6

Some Transformation, Reduction And Summation Formulae Involving Hypergeometric Polynomials-II

6.1 Introduction

The main object of the present chapter is to establish another five new transformations for Srivastava-Daoust double hypergeometric polynomials with $-1, +1$ arguments using Beta integral technique in the evaluation of the integral $\int_{-1}^{+1} (1+x)^b (1-x)^c P_n^{(\alpha, \beta)}(x) dx$ by different approaches. Along with it, A summation theorem for hypergeometric polynomials ${}_5F_4(1)$, some interesting special cases of main integrals and reduction formulae for hypergeometric polynomials ${}_5F_4(-1)$ in terms of terminating series ${}_3F_2(1)$ and truncated series ${}_2F_1(1)$, are also deduced.

The Jacobi Polynomials of first kind $P_n^{(\alpha, \beta)}(x)$ is defined by the following one more equation [85, p.275(Q.5)]:

$$P_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{(1+\alpha+\beta)_n} \sum_{k=0}^n \frac{(-1)^{n-k} (\beta)_{n-k} (1+\alpha+\beta)_{n+k} (1+\alpha+2k)}{(n-k)!(1+\alpha)_{n+k+1}} P_k^{(\alpha, 0)}(x) \quad (6.1.1)$$

where n is a non-negative integer.

Terminating hypergeometric series

Terminating hypergeometric series is defined by

$${}_pF_q \left[\begin{matrix} -n, a_2, \dots, a_p; \\ b_1, b_2, \dots, b_q; \end{matrix} z \right] = \sum_{r=0}^n \frac{(-n)_r (a_2)_r \dots (a_p)_r z^r}{(b_1)_r (b_2)_r \dots (b_q)_r r!}, \quad (6.1.2)$$

where n is non-negative integer and $(a_2, \dots, a_p, b_1, \dots, b_q \in \mathbb{C} \setminus \mathbb{Z}_0^-)$.

Truncated hypergeometric series

Truncated hypergeometric series, denoted by ${}_pF_q [z]_n$, is defined as

$${}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right]_n = \sum_{r=0}^n \frac{(\alpha_1)_r (\alpha_2)_r \dots (\alpha_p)_r z^r}{(\beta_1)_r (\beta_2)_r \dots (\beta_q)_r r!}; \quad (6.1.3)$$

$$(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

When $n \rightarrow \infty$, it reduces to the non-terminating hypergeometric series (1.3.3).

6.2 Transformations For Srivastava-Daoust Double Hypergeometric Polynomials

If n is non-negative integer and any values of parameters leading to the results which do not make sense, are tacitly excluded, then

$$\begin{aligned}
 & F_{3: 0; 2}^{4: 0; 1} \left(\begin{array}{l} [-n : 1, 1], [\frac{\alpha+3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [\alpha + 1 : 1, 2] : \\ [\frac{\alpha+1}{2} : 1, 1], [1 - \beta - n : 1, 1], [\alpha + n + 2 : 1, 1] \end{array} : \right. \\
 & \qquad \qquad \qquad \left. \begin{array}{l} \text{---}; \qquad [c + 1 : 1] \qquad ; \\ \text{---}; [1 + \alpha : 1], [c + b + 2 : 1]; \end{array} \right. \begin{array}{l} \\ -1, 1 \end{array} \Bigg) \\
 & = F_{4: 1; 1}^{5: 1; 1} \left(\begin{array}{l} [-n : 1, 1], [\frac{\alpha+3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [\alpha + 1 : 1, 1], [1 : 1, 1] : \\ [\frac{\alpha+1}{2} : 1, 1], [1 - \beta - n : 1, 1], [\alpha + n + 2 : 1, 1], [c + b + 2 : 1, 1] \end{array} : \right. \\
 & \qquad \qquad \qquad \left. \begin{array}{l} [b + 1 : 1]; [c + 1 : 1]; \\ [1 : 1]; [\alpha + 1 : 1] \end{array} ; \right. \begin{array}{l} \\ -1, 1 \end{array} \Bigg) \qquad (6.2.1)
 \end{aligned}$$

$$\begin{aligned}
 & = F_{4: 0; 2}^{5: 0; 1} \left(\begin{array}{l} [-n : 1, 1], [\frac{\alpha+3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [\alpha + 1 : 1, 2], [1 : 1, 1] : \\ [\frac{\alpha+1}{2} : 1, 1], [1 - \beta - n : 1, 1], [\alpha + n + 2 : 1, 1], [\alpha + 1 : 1, 1] \end{array} : \right. \\
 & \qquad \qquad \qquad \left. \begin{array}{l} \text{---}; \qquad [b + 1 : 1] \qquad ; \\ \text{---}; [1 : 1], [c + b + 2 : 1]; \end{array} \right. \begin{array}{l} \\ 1, -1 \end{array} \Bigg) \qquad (6.2.2)
 \end{aligned}$$

$$= F_{3: 2; 0}^{5: 1; 0} \left(\begin{array}{l} [-n : 1, 1], [\frac{\alpha+3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [\alpha + 1 : 2, 1], [\alpha + 1 : 1, 1] : \\ [\frac{\alpha+1}{2} : 1, 1], [1 - \beta - n : 1, 1], [\alpha + n + 2 : 1, 1] \end{array} : \right.$$

where n is non-negative integer and remaining numerator and denominator parameters are neither zero nor negative integers.

6.4 Evaluation of the integral $\int_{-1}^{+1} (1+x)^b (1-x)^c P_n^{(\alpha, \beta)}(x) dx$ using the definition of $P_k^{(\alpha, 0)}(x)$

If any values of parameters leading to the results which do not make sense, are tacitly excluded, then

$$\begin{aligned} & \{\psi_n(b, c, \alpha, \beta)\}^{-1} \int_{-1}^{+1} (1+x)^b (1-x)^c P_n^{(\alpha, \beta)}(x) dx \\ &= \sum_{k=0}^n \frac{(-1)^k (-n)_k (\alpha+1)_k \left(\frac{\alpha+3}{2}\right)_k (\alpha+\beta+n+1)_k}{k! (\alpha+n+2)_k \left(\frac{\alpha+1}{2}\right)_k (1-\beta-n)_k} {}_3F_2 \left[\begin{matrix} -k, \alpha+k+1, c+1; \\ 1+\alpha, c+b+2; \end{matrix} \middle| 1 \right] \end{aligned} \quad (6.4.1)$$

$$\begin{aligned} &= F \begin{matrix} 4: 0; 1 \\ 3: 0; 2 \end{matrix} \left(\begin{matrix} [-n : 1, 1], [\frac{\alpha+3}{2} : 1, 1], [\alpha+\beta+n+1 : 1, 1], [\alpha+1 : 1, 2] : \\ [\frac{\alpha+1}{2} : 1, 1], [1-\beta-n : 1, 1], [\alpha+n+2 : 1, 1] : \\ \text{---}; [c+1 : 1] ; \\ \text{---}; [1+\alpha : 1], [c+b+2 : 1] ; \end{matrix} \begin{matrix} -1, 1 \end{matrix} \right) \end{aligned} \quad (6.4.2)$$

$$\begin{aligned} &= \sum_{k=0}^n \frac{(-1)^k (-n)_k (\alpha+1)_k (\alpha+\beta+n+1)_k}{k! (\alpha+n+2)_k (1-\beta-n)_k} \frac{\left(\frac{\alpha+3}{2}\right)_k (b+1)_k}{\left(\frac{\alpha+1}{2}\right)_k (b+c+2)_k} \times \\ & \quad \times {}_3F_2 \left[\begin{matrix} -k, -k, c+1; \\ 1+\alpha, -b-k; \end{matrix} \middle| 1 \right] \end{aligned} \quad (6.4.3)$$

$$= F \begin{matrix} 5: 1; 1 \\ 4: 1; 1 \end{matrix} \left(\begin{matrix} [-n : 1, 1], [\frac{\alpha+3}{2} : 1, 1], [\alpha+\beta+n+1 : 1, 1], [\alpha+1 : 1, 1], \\ [\frac{\alpha+1}{2} : 1, 1], [1-\beta-n : 1, 1], [\alpha+n+2 : 1, 1], \end{matrix} \right)$$

$$\left(\begin{array}{c} [1 : 1, 1] \quad : [b + 1 : 1]; [c + 1 : 1]; \\ \\ [c + b + 2 : 1, 1] : [1 : 1]; [\alpha + 1 : 1]; \end{array} \quad \begin{array}{c} -1, 1 \end{array} \right) \quad (6.4.4)$$

$$= \sum_{k=0}^n \frac{(-n)_k (\alpha + \beta + n + 1)_k \left(\frac{\alpha+3}{2}\right)_k}{(\alpha + n + 2)_k (1 - \beta - n)_k \left(\frac{\alpha+1}{2}\right)_k} {}_3F_2 \left[\begin{array}{c} -k, \alpha + k + 1, b + 1; \\ 1, \quad c + b + 2; \end{array} \quad 1 \right] \quad (6.4.5)$$

$$= F_{4: 0; 2}^{5: 0; 1} \left(\begin{array}{c} [-n : 1, 1], \left[\frac{\alpha+3}{2} : 1, 1\right], [\alpha + \beta + n + 1 : 1, 1], [\alpha + 1 : 1, 2], \\ \\ \left[\frac{\alpha+1}{2} : 1, 1\right], [1 - \beta - n : 1, 1], [\alpha + n + 2 : 1, 1], [\alpha + 1 : 1, 1] \\ \\ [1 : 1, 1] : \text{---}; \quad [b + 1 : 1] \quad ; \\ \\ 1, -1 \\ \\ : \text{---}; [1 : 1], [c + b + 2 : 1]; \end{array} \right) \quad (6.4.6)$$

$$= \sum_{k=0}^n \frac{(-n)_k (\alpha + \beta + n + 1)_k \left(\frac{\alpha+3}{2}\right)_k \left(\frac{\alpha+2}{2}\right)_k (c + 1)_k 4^k}{k! (\alpha + n + 2)_k (1 - \beta - n)_k (b + c + 2)_k} \times \\ \times {}_3F_2 \left[\begin{array}{c} -k, -\alpha - k, -b - c - k - 1; \\ -\alpha - 2k, \quad -c - k; \end{array} \quad 1 \right] \quad (6.4.7)$$

$$= F_{3: 2; 0}^{5: 1; 0} \left(\begin{array}{c} [-n : 1, 1], \left[\frac{\alpha+3}{2} : 1, 1\right], [\alpha + \beta + n + 1 : 1, 1], [\alpha + 1 : 2, 1], \\ \\ \left[\frac{\alpha+1}{2} : 1, 1\right], [1 - \beta - n : 1, 1], [\alpha + n + 2 : 1, 1] \\ \\ [\alpha + 1 : 1, 1] : \quad [c + 1 : 1]; \text{---}; \\ \\ 1, -1 \\ \\ : [1 + \alpha : 1], [c + b + 2 : 1]; \text{---}; \end{array} \right) \quad (6.4.8)$$

$$= \sum_{k=0}^n \frac{(-n)_k (\alpha + \beta + n + 1)_k \left(\frac{\alpha+3}{2}\right)_k (c + 1)_k}{(\alpha + n + 2)_k (1 - \beta - n)_k \left(\frac{\alpha+1}{2}\right)_k (b + c + 2)_k} {}_3F_2 \left[\begin{array}{c} -k, -\alpha - k, b + 1; \\ 1, \quad -c - k; \end{array} \quad 1 \right] \quad (6.4.9)$$

$$= F_{4: 1; 1}^{5: 1; 1} \left(\begin{array}{l} [-n : 1, 1], [\frac{\alpha+3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [1 : 1, 1], [\alpha + 1 : 1, 1] : \\ [\frac{\alpha+1}{2} : 1, 1], [1 - \beta - n : 1, 1], [\alpha + n + 2 : 1, 1], [c + b + 2 : 1, 1] : \\ [c + 1 : 1]; [b + 1 : 1]; \\ 1, -1 \\ [\alpha + 1 : 1]; [1 : 1] \end{array} \right) \quad (6.4.10)$$

$$= \sum_{k=0}^n \frac{(-1)^k (-n)_k (\alpha + \beta + n + 1)_k (\frac{\alpha+3}{2})_k (\frac{\alpha+2}{2})_k (b+1)_k 4^k}{k! (\alpha + n + 2)_k (1 - \beta - n)_k (b + c + 2)_k (\alpha + 1)_k} \times \\ \times {}_3F_2 \left[\begin{array}{l} -k, -k, -b - c - k - 1; \\ -\alpha - 2k, -b - k; \end{array} \begin{array}{l} 1 \\ 1 \end{array} \right] \quad (6.4.11)$$

$$= F_{4: 2; 0}^{5: 1; 0} \left(\begin{array}{l} [-n : 1, 1], [\frac{\alpha+3}{2} : 1, 1], [\alpha + \beta + n + 1 : 1, 1], [\alpha + 1 : 2, 1], [1 : 1, 1] : \\ [\frac{\alpha+1}{2} : 1, 1], [1 - \beta - n : 1, 1], [\alpha + n + 2 : 1, 1], [\alpha + 1 : 1, 1] : \\ [b + 1 : 1] ; \text{---}; \\ -1, 1 \\ [1 : 1], [c + b + 2 : 1] ; \text{---}; \end{array} \right) \quad (6.4.12)$$

where

$$\psi_n(b, c, \alpha, \beta) = \frac{(-1)^n 2^{c+b+1} \Gamma(b+1) \Gamma(c+1) (1+\alpha)_n (\beta)_n}{n! \Gamma(c+b+2) (\alpha+2)_n} \\ (\Re(b) > -1, \Re(c) > -1).$$

Proof:- Applying first, the definition (6.1.1) for $P_n^{(\alpha, \beta)}(x)$ and then definition (5.1.1) for Jacobi polynomials $P_k^{(\alpha, 0)}(x)$ in the integral $\int_{-1}^{+1} (1+x)^b (1-x)^c P_n^{(\alpha, \beta)}(x) dx$, expanding hypergeometric polynomial ${}_2F_1$ in series form, interchanging the order of integration and summation, using the Beta integral (5.1.8), we get the result (6.4.1) and applying the summation property (5.1.11) in the result (6.4.1) then writing the resulting double series in Srivastava-Daoust notation, we get the result (6.4.2).

Similarly, after using the definition (6.1.1) in $P_n^{(\alpha, \beta)}(x)$ in the integral

$$\int_{-1}^{+1} (1+x)^b (1-x)^c P_n^{(\alpha, \beta)}(x) dx$$

and then using the remaining five definitions (5.1.2), (5.1.3), (5.1.4), (5.1.5) and (5.1.6) of Jacobi polynomials in $P_k^{(\alpha,0)}(x)$ in place of the definition (5.1.1), we can evaluate the same integral in the forms of (6.4.3), (6.4.4); (6.4.5), (6.4.6); (6.4.7), (6.4.8); (6.4.9), (6.4.10) and (6.4.11), (6.4.12) respectively.

6.5 Special Cases of Main Integrals

Case (i):- Putting $b = 0$ in equations (6.4.1) and (6.4.7), applying Saalschütz' summation theorem (5.1.10) and putting $b = 0$ in equations (6.4.3), (6.4.5), (6.4.9) and (6.4.11), applying Gauss theorem (5.1.9), we get

$$\begin{aligned} {}_5F_4 \left[\begin{matrix} -n, 1, \alpha - c, \frac{\alpha+3}{2}, \alpha + \beta + n + 1 \\ \alpha + n + 2, c + 2, \frac{\alpha+1}{2}, 1 - \beta - n \end{matrix} ; -1 \right] \\ = \frac{(-1)^n n!(c+1)(\alpha+n+1)}{2^{c+1}(\alpha+1)(\beta)_n} \int_{-1}^{+1} (1-x)^c P_n^{(\alpha,\beta)}(x) dx \quad (6.5.1) \\ (\Re(c) > -1). \end{aligned}$$

Case (ii):- Putting $c = \alpha$ in equations (6.4.1), (6.4.3), (6.4.7) and (6.4.9), applying Gauss theorem (5.1.9) and putting $c = \alpha$ in equations (6.4.5) and (6.4.11), applying Saalschütz' summation theorem (5.1.10), we get

$$\begin{aligned} \int_{-1}^{+1} (1+x)^b (1-x)^\alpha P_n^{(\alpha,\beta)}(x) dx = \frac{(-1)^n 2^{\alpha+b+1} \Gamma(b+1) \Gamma(\alpha+n+1) (\beta)_n}{\Gamma(\alpha+b+2) (\alpha+2)_n n!} \times \\ \times {}_5F_4 \left[\begin{matrix} -n, -b, \alpha + 1, \frac{\alpha+3}{2}, \alpha + \beta + n + 1 \\ \alpha + b + 2, \alpha + n + 2, \frac{\alpha+1}{2}, 1 - \beta - n \end{matrix} ; 1 \right] \quad (6.5.2) \\ (\Re(\alpha) > -1, \Re(b) > -1). \end{aligned}$$

6.6 Reduction Formulae For Terminating Series

$${}_5F_4(-1)$$

If n is non-negative integer and remaining numerator, denominator parameters are neither zero nor negative integers, then

$$\begin{aligned} & {}_5F_4 \left[\begin{matrix} -n, \alpha - c, 1, \frac{\alpha+3}{2}, \alpha + \beta + n + 1 \\ c + 2, \alpha + n + 2, \frac{\alpha+1}{2}, 1 - \beta - n \end{matrix} ; -1 \right] \\ &= \frac{(-1)^n (\alpha + 2)_n}{(\beta)_n} {}_3F_2 \left[\begin{matrix} -n, \alpha + \beta + n + 1, c + 1; \\ \alpha + 1, c + 2 \end{matrix} ; 1 \right] \end{aligned} \quad (6.6.1)$$

$$= \frac{(\alpha + n + 1) (1 + \beta)_n}{(1 + \alpha) (\beta)_n} {}_3F_2 \left[\begin{matrix} -n, \alpha + \beta + n + 1, 1; \\ \beta + 1, c + 2 \end{matrix} ; 1 \right] \quad (6.6.2)$$

$$= \frac{(\alpha + n + 1) (c + 1) (1 + \alpha + \beta)_{2n}}{(\beta)_n (1 + \alpha + \beta)_n (c + n + 1) (1 + \alpha)} {}_3F_2 \left[\begin{matrix} -n, -\alpha - n, -c - n - 1; \\ -c - n, -\alpha - \beta - 2n \end{matrix} ; 1 \right] \quad (6.6.3)$$

$$= \frac{(c + 1) (\alpha + n + 1) (1 + \beta)_n}{(\alpha + 1) (c + n + 1) (\beta)_n} {}_3F_2 \left[\begin{matrix} -n, -\alpha - n, 1; \\ 1 + \beta, -c - n \end{matrix} ; 1 \right] \quad (6.6.4)$$

Proof:- If we solve the integral in the right hand side of equation (6.5.1) using the definitions (5.1.1), (5.1.3), (5.1.4) and (5.1.5) for $P_n^{(\alpha, \beta)}(x)$ with the help of the Beta integral (5.1.8), then we get four reduction formulae (6.6.1), (6.6.2), (6.6.3) and (6.6.4) for hypergeometric polynomials ${}_5F_4(-1)$ in terms of hypergeometric polynomials ${}_3F_2(1)$ respectively.

Remark:- It should be noted that we cannot find the sum of above hypergeometric polynomials ${}_3F_2$ in terms of gamma functions, using Saalschütz' theorem (5.1.10), Watson's theorem (5.1.16), Dixon's theorem (5.1.17), Whipple's theorem (5.1.19) and other summation theorems available in the literature of Special functions.

6.7 Reduction Formulae Involving Terminating Series ${}_5F_4(-1)$ and Truncated Series ${}_2F_1(1)$

$$\begin{aligned}
 & {}_5F_4 \left[\begin{matrix} -n, \alpha - c, 1, \frac{\alpha+3}{2}, \alpha + \beta + n + 1 & ; & -1 \\ c + 2, \alpha + n + 2, \frac{\alpha+1}{2}, 1 - \beta - n & ; & \end{matrix} \right] \\
 &= \frac{(-1)^n n! (\alpha + 2)_n}{(\beta)_n (c + 2)_n} {}_2F_1 \left[\begin{matrix} -\beta - n, c + 1 & ; & 1 \\ 1 + \alpha & ; & \end{matrix} \right]_n \tag{6.7.1}
 \end{aligned}$$

$$= \frac{(-1)^n n! (\alpha + n + 1) (1 + \alpha + \beta)_{2n}}{(\beta)_n (c + 2)_n (\alpha + 1) (1 + \alpha + \beta)_n} {}_2F_1 \left[\begin{matrix} -\beta - n, -c - n - 1 & ; & 1 \\ -\alpha - \beta - 2n & ; & \end{matrix} \right]_n \tag{6.7.2}$$

Proof:- We can get the above two results using definitions (5.1.2) and (5.1.6) of Jacobi polynomials $P_n^{\alpha, \beta}(x)$ in the right hand side of equation (6.5.1).

Chapter 7

Some Generating Relations And Hypergeometric Transformations Using Laplace And Beta Integral Techniques

7.1 Introduction

Using Laplace transform technique, disclosure of some interesting generating relations associated with double hypergeometric function H_4 of Horn, triple hypergeometric functions X_2, X_7 of Exton, F_E, F_G of Saran and quadruple hypergeometric functions K_9, K_{12}, K_{13} of Exton have been made in this chapter and also obtained the correct form of two erroneous integrals of Erdélyi et al. [*Tables of Integral Transforms*, Vol.II, p.284(16.4.1 and 16.4.3)]. Five hypergeometric transformations of the terminating Clausen's series ${}_3F_2$ are uncovered too.

If function (continuous or discontinuous) $f(t)$ is well defined for $t > 0$ and $\lim_{t \rightarrow \infty} e^{-bt} f(t) = a$ finite quantity, then operational images (or operational representation) of many classes of Special Functions is the classical Laplace transform

$$L\{f(t) : s\} = \int_0^{\infty} e^{-st} f(t) dt = F(s), \quad (7.1.1)$$

(where $Re(s) > b > 0$ and e^{-st} is called kernel.)

can be obtained by appealing to Eulerian integral of second kind [117, p.218(3)]

$$L\{t^{A-1} : s\} = \int_0^{\infty} e^{-st} t^{A-1} dt = \frac{\Gamma(A)}{s^A}; \min\{Re(A), Re(s)\} > 0, \quad (7.1.2)$$

where the notation (Γ) is used to denote Gamma.

For the sake of convenience in the derivations of our generating relations, we shall recall the power series forms of different multiple Gaussian hypergeometric functions:

$$\Phi_2[a, b; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_n x^m y^n}{(c)_{m+n} m! n!} \quad (7.1.3)$$

($|x| < \infty, |y| < \infty$)

$$\Psi_2[a; b, c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} x^m y^n}{(b)_m (c)_n m! n!} \quad (7.1.4)$$

($|x| < \infty, |y| < \infty$)

$$H_4[a, b; c, d; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n} (b)_n x^m y^n}{(c)_m (d)_n m! n!} \quad (7.1.5)$$

($|x| < r, |y| < s; 4r = (s-1)^2$)

[37, 1976, p.97(3.5.2)]

$${}^{(2)}H_4^{(3)} : X_2 [a, b; c, d, e; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+2n+p} (b)_p x^m y^n z^p}{(c)_m (d)_n (e)_p m! n! p!} \quad (7.1.6)$$

$$\begin{aligned} & \left(2(\sqrt{|x|} + \sqrt{|y|}) + |z| < 1 \right), \\ X_7 [a, b, c; d, e; x, y, z] &= \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p} (b)_n (c)_p x^m y^n z^p}{(d)_m (e)_{n+p} m! n! p!} \quad (7.1.7) \\ & (|4x| < 1, |y| < 1, |z| < 1) \end{aligned}$$

$$F_4 : F_E [a, a, a, b, c, c; d, e, g; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p} (b)_m (c)_{n+p} x^m y^n z^p}{(d)_m (e)_n (g)_p m! n! p!} \quad (7.1.8)$$

$$(|x| < r, |y| < s, |z| < t; r + (\sqrt{s} + \sqrt{t})^2 = 1)$$

$$F_8 : F_G [a, a, a, b, c, d; e, f, f; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p} (b)_m (c)_n (d)_p x^m y^n z^p}{(e)_m (f)_{n+p} m! n! p!} \quad (7.1.9)$$

$$(|x| < r, |y| < s, |z| < t; r + s = 1 = r + t)$$

$$\begin{aligned} & K_9 [a, a, a, a; b, b, c, d; e, f, g, g; x, y, z, t] \\ &= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q} (b)_{m+n} (c)_p (d)_q x^m y^n z^p t^q}{(e)_m (f)_n (g)_{p+q} m! n! p! q!} \quad (7.1.10) \end{aligned}$$

$$\begin{aligned} & {}^{(2)}_{(1)}E_D^{(4)} : K_{12} [a, a, a, a; b, c, d, e; f, f, g, g; x, y, z, t] \\ &= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q} (b)_m (c)_n (d)_p (e)_q x^m y^n z^p t^q}{(f)_{m+n} (g)_{p+q} m! n! p! q!} \quad (7.1.11) \end{aligned}$$

$$\begin{aligned} & K_{13} [a, a, a, a; b, c, d, e; f, f, g, h; x, y, z, t] \\ &= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q} (b)_m (c)_n (d)_p (e)_q x^m y^n z^p t^q}{(f)_{m+n} (g)_p (h)_q m! n! p! q!} \quad (7.1.12) \end{aligned}$$

where the Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) is defined, in terms of the familiar Gamma function, by equation (1.2.11). Denominator parameters are neither zero nor negative integers, numerator parameters may be zero or negative integers.

For general theory of convergence of Exton's quadruple hypergeometric functions, see [37, pp.91 – 94].

Here Φ_2, Ψ_2 are double hypergeometric functions of Humbert [55; see also 117, p.58 (37), p.59(42)]; H_4 is a double hypergeometric function of Horn [51; see also 117, p.57 (28)]; X_2, X_7 are triple hypergeometric functions of Exton [40, p.113 (1.2, 1.7); see also 116]; F_E, F_G are triple hypergeometric functions of Saran [117, p.66 (26),

$p.67$ (28); see also [88, 91]; K_9, K_{12}, K_{13} are quadruple hypergeometric functions of Exton [30, 32, 35]. In another notation of Exton, X_2 is also denoted by ${}^{(2)}H_4^{(3)}$ [37, p.97(3.5.2)]; in the notation of Lauricella [70], Saran's functions F_E and F_G are also denoted by F_4 and F_8 respectively; in another notation of Exton, K_{12} is also denoted by ${}^{(2)}E_D^{(4)}$ [37, p.89 (3.4.1); see also 31, 34, 35].

Two definite integrals of Erdélyi et al. [26, p.284(16.4.1, 16.4.3)]

$$\int_{-1}^{+1} (1-x)^\alpha (1+x)^\sigma P_n^{(\alpha, \beta)}(x) dx = \frac{2^{\alpha+\sigma+1} \Gamma(\sigma+1) \Gamma(\alpha+n+1) \Gamma(\sigma-\beta+1)}{\Gamma(\sigma-\beta-n+1) \Gamma(\alpha+\sigma+n+2)} \quad (7.1.13)$$

where $Re(\alpha) > -1$, $Re(\sigma) > -1$.

$$\begin{aligned} \int_{-1}^{+1} (1-x)^\rho (1+x)^\sigma P_n^{(\alpha, \beta)}(x) dx &= \frac{2^{\rho+\sigma+1} \Gamma(\rho+1) \Gamma(\sigma+1)}{\Gamma(\rho+\sigma+2)} \times \\ &\times {}_3F_2(-n, \alpha+\beta+n+1, \rho+1; \alpha+1, \rho+\sigma+2; 1) \end{aligned} \quad (7.1.14)$$

where $Re(\rho) > -1$, $Re(\sigma) > -1$.

Combinatorial identity

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = \sum_{k=0}^n \frac{(-n)_k}{k!} = \begin{cases} 0, & \text{if } n = 1, 2, 3, \dots \\ 1, & \text{if } n=0 \end{cases} \quad (7.1.15)$$

which can be verified easily and $\binom{n}{k}$ is binomial coefficient.

An important property of Jacobi polynomials [85, p.256(133.2)]

$$P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x) \quad (7.1.16)$$

Definite integral property

$$\int_{-1}^{+1} F(x) dx = \int_{-1}^{+1} F(-x) dx \quad (7.1.17)$$

7.2 Main Generating Relations

Arguments and parameters are adjusted in such a way that each term of involved multiple power series is completely well-defined and meaningful then without any loss of convergence, we have:

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(a)_n}{n!} H_4 \left[g, b+m+n; d, b; \frac{y}{(1+x)^2}, \frac{x}{(1+x)} \right] t^n \\
&= \frac{(1+x)^g}{(1-t)^a} X_7 \left[g, -m, a; d, b; y, -x, \frac{xt}{(1-t)} \right], \tag{7.2.1}
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(a)_n}{n!} F_G \left[g, g, g, b+m+n, c, d; b, h, h; \frac{3x}{x-3}, \frac{3z}{3-x}, \frac{3w}{3-x} \right] t^n \\
&= (1-t)^{-a} \left(\frac{3-x}{3+2x} \right)^g K_{12} \left[g, g, g, g; c, d, -m, a; h, h, b, b; \right. \\
&\quad \left. \frac{3z}{3+2x}, \frac{3w}{3+2x}, \frac{3x}{3+2x}, \frac{3xt}{(t-1)(3+2x)} \right], \tag{7.2.2}
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(a)_n}{n!} X_2 \left[g, -m-n; c, d, b; \left(\frac{y}{1-2y} \right)^2, \left(\frac{z}{1-2y} \right)^2, \frac{x}{1-2y} \right] t^n \\
&= (1-t)^{-a} \left(\frac{1-2y}{1+2z} \right)^g K_{13} \left[g, g, g, g; -m, a, c - \frac{1}{2}, d - \frac{1}{2}; b, b, 2c-1, 2d-1; \right. \\
&\quad \left. \frac{x}{1+2z}, \frac{-xt}{(1+2z)(1-t)}, \frac{4y}{1+2z}, \frac{4z}{1+2z} \right], \tag{7.2.3}
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(a)_n}{n!} F_E \left[\beta, \beta, \beta, b+m+n, c, c; b, g, h; \frac{x}{\lambda+\mu x}, \frac{y}{\lambda+\mu x}, \frac{z}{\lambda+\mu x} \right] t^n = (1-t)^{-a} \\
& \left(\frac{\lambda+\mu x}{\lambda+\mu x-x} \right)^\beta K_9 \left[\beta, \beta, \beta, \beta; c, c, a, -m; g, h, b, b; \frac{y}{\lambda+\mu x-x}, \frac{z}{\lambda+\mu x-x}, \right. \\
& \quad \left. \frac{xt}{(1-t)(\lambda+\mu x-x)}, \frac{x}{x-\mu x-\lambda} \right]. \tag{7.2.4}
\end{aligned}$$

7.3 Derivations

Consider the generating relation of Manocha [75, p.225 (3); see also 117, p.132 (6), p.268 (12i)] in the form:

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} {}_1F_1 \left[\begin{matrix} -m-n; \\ b \end{matrix}; x \right] t^n = (1-t)^{-a} \cdot \Phi_2 \left[-m, a; b; x, \frac{-xt}{1-t} \right]$$

$$\text{where } m = 0, 1, 2, \dots \text{ and } |t| < 1. \tag{7.3.1}$$

If we use Kummer's first transformation [85, p.125 (42); 117 p.37 (7)]

$${}_1F_1 \left[\begin{matrix} a; \\ b; \end{matrix} x \right] = e^x {}_1F_1 \left[\begin{matrix} b-a; \\ b; \end{matrix} -x \right],$$

(provided b is neither zero nor a negative integer.) for ${}_1F_1$ in (7.3.1), we get

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} e^x {}_1F_1 \left[\begin{matrix} b+m+n; \\ b; \end{matrix} -x \right] t^n = (1-t)^{-a} \cdot \Phi_2 \left[-m, a; b; x, \frac{-xt}{1-t} \right] \quad (7.3.2)$$

In (7.3.2), replacing x by $-xu$, multiplying both sides by $e^{-u} u^{g-1} {}_0F_1 \left[\begin{matrix} -; \\ d; \end{matrix} yu^2 \right]$, expressing different hypergeometric functions in power series forms and integrating with respect to u over the interval $(0, \infty)$, after simplification we get (7.2.1).

If we write (7.3.2) in the following form after replacing x by xu :

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} e^{xu/3} {}_1F_1 \left[\begin{matrix} b+m+n; \\ b; \end{matrix} -xu \right] t^n = (1-t)^{-a} \cdot e^{-2xu/3} \cdot \Phi_2 \left[-m, a; b; xu, \frac{-xut}{1-t} \right] \quad (7.3.3)$$

and multiply (7.3.3) by $e^{-u} u^{g-1} \Phi_2[c, d; h; zu, wu]$, express different hypergeometric functions in power series forms and integrate with respect to u over the interval $(0, \infty)$, we get (7.2.2).

In (7.3.1), replacing x by xu , multiplying left hand side by

$$e^{-u(1-2y)} u^{g-1} {}_0F_1 \left[\begin{matrix} -; \\ c; \end{matrix} y^2 u^2 \right] {}_0F_1 \left[\begin{matrix} -; \\ d; \end{matrix} z^2 u^2 \right]$$

and right hand side by

$$e^{-u(1+2z)} u^{g-1} {}_1F_1 \left[\begin{matrix} \frac{2c-1}{2}; \\ 2c-1; \end{matrix} 4yu \right] {}_1F_1 \left[\begin{matrix} \frac{2d-1}{2}; \\ 2d-1; \end{matrix} 4zu \right],$$

using Kummer's second transformation [85, p.126 (43); 117, p.37 (8)]

$$e^{-x} {}_1F_1 \left[\begin{matrix} a; \\ 2a; \end{matrix} 2x \right] = {}_0F_1 \left[\begin{matrix} -; \\ a + \frac{1}{2}; \end{matrix} \frac{x^2}{4} \right],$$

where

$$2a \neq -1, -3, -5, \dots \text{ (i.e. negative odd integers)}$$

expressing different hypergeometric functions in power series forms, integrating with respect to u over the interval $(0, \infty)$, we get (7.2.3).

If we write (7.3.2) in the following form after replacing x by $-xu$:

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} {}_1F_1 \left[\begin{matrix} b+m+n; \\ b \end{matrix}; xu \right] t^n = (1-t)^{-a} e^{xu} \Phi_2 \left[-m, a; b; -xu, \frac{uxt}{1-t} \right] \quad (7.3.4)$$

and multiply both sides of (7.3.4) by $e^{-(\lambda+\mu x)u} u^{\beta-1} \Psi_2 [c; g, h; yu, zu]$, express different hypergeometric functions in power series forms and integrate with respect to u over the interval $(0, \infty)$, after simplification we get (7.2.4).

7.4 Hypergeometric Transformations Involving Terminating Clausen's Series Using Beta Integral Technique

7.4.1 Evaluation of the integral $\int_{-1}^{+1} (1-x)^a (1+x)^b P_n^{(\alpha, \beta)}(x) dx$

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\begin{aligned} & \int_{-1}^{+1} (1-x)^a (1+x)^b P_n^{(\alpha, \beta)}(x) dx \\ &= \frac{2^{a+b+1} \Gamma(b+1) \Gamma(a+1)}{n!} \frac{(1+\alpha)_n}{\Gamma(a+b+2)} {}_3F_2 \left[\begin{matrix} -n, \alpha + \beta + n + 1, a + 1; \\ 1 + \alpha, \quad a + b + 2; \end{matrix} 1 \right] \end{aligned} \quad (7.4.1)$$

$$= \frac{2^{a+b+1} \Gamma(a+1) \Gamma(b+n+1) (1+\alpha)_n}{n! \Gamma(a+b+n+2)} {}_3F_2 \left[\begin{matrix} -n, -\beta - n, a + 1; \\ 1 + \alpha, \quad -b - n; \end{matrix} 1 \right] \quad (7.4.2)$$

$$= \frac{2^{a+b+1} \Gamma(a+1) \Gamma(b+1) (-1)^n (1+\beta)_n}{n! \Gamma(a+b+2)} {}_3F_2 \left[\begin{matrix} -n, 1 + \alpha + \beta + n, b + 1; \\ 1 + \beta, \quad a + b + 2; \end{matrix} 1 \right] \quad (7.4.3)$$

$$= \frac{2^{a+b+1} (-1)^n \Gamma(b+1) \Gamma(a+n+1) (1+\alpha+\beta)_{2n}}{n! (1+\alpha+\beta)_n \Gamma(a+b+n+2)}.$$

$${}_3F_2 \left[\begin{matrix} -n, -\alpha - n, -a - b - n - 1; \\ -\alpha - \beta - 2n, \quad -a - n; \end{matrix} 1 \right] \quad (7.4.4)$$

$$= \frac{(-1)^n 2^{a+b+1} (1+\beta)_n \Gamma(b+1) \Gamma(a+n+1)}{n! \Gamma(a+b+n+2)} {}_3F_2 \left[\begin{matrix} -n, -\alpha - n, b+1; \\ 1+\beta, \quad -a - n; \end{matrix} 1 \right] \quad (7.4.5)$$

$$= \frac{2^{a+b+1} \Gamma(a+1) \Gamma(b+n+1) (1+\alpha+\beta)_{2n}}{n! (1+\alpha+\beta)_n \Gamma(a+b+n+2)} {}_3F_2 \left[\begin{matrix} -n, -\beta - n, -a - b - n - 1; \\ -\alpha - \beta - 2n, \quad -b - n; \end{matrix} 1 \right] \quad (7.4.6)$$

where $Re(a) > -1$, $Re(b) > -1$.

Proof Applying the definition (5.1.1) of Jacobi polynomials in the integral $\int_{-1}^{+1} (1-x)^a (1+x)^b P_n^{(\alpha, \beta)}(x) dx$, expanding hypergeometric polynomial ${}_2F_1$ in series form, interchanging the order of integration and summation, using the Beta integral (5.1.8) and then writing the resulting series in hypergeometric notation, we get result (7.4.1).

Similarly using the remaining five definitions (5.1.2), (5.1.3), (5.1.4), (5.1.5) and (5.1.6) of Jacobi polynomials, we can evaluate the same integral in the forms of (7.4.2), (7.4.3), (7.4.4), (7.4.5) and (7.4.6) respectively.

7.5 Main Hypergeometric Transformations

If any values of parameters leading to the results which do not make sense, are tacitly excluded, then

$${}_3F_2 \left[\begin{matrix} -n, 1+\alpha+\beta+n, 1+a; \\ 1+\alpha, \quad a+b+2 \end{matrix} ; 1 \right] \\ = \frac{(1+b)_n}{(a+b+2)_n} {}_3F_2 \left[\begin{matrix} -n, -\beta - n, 1+a \\ 1+\alpha, -b-n \end{matrix} ; 1 \right] \quad (7.5.1)$$

$$= \frac{(-1)^n (1+\beta)_n}{(1+\alpha)_n} {}_3F_2 \left[\begin{matrix} -n, 1+b, 1+\alpha+\beta+n; \\ 1+\beta, a+b+2 \end{matrix} ; 1 \right] \quad (7.5.2)$$

$$= \frac{(-1)^n (1+a)_n \Gamma(1+\alpha+\beta+2n)}{(1+\alpha)_n \Gamma(1+\alpha+\beta+n)(a+b+2)_n} {}_3F_2 \left[\begin{matrix} -n, -\alpha-n, -a-b-n-1; \\ -a-n, -\alpha-\beta-2n \end{matrix} ; 1 \right] \quad (7.5.3)$$

$$= \frac{(-1)^n (1+\beta)_n (1+a)_n}{(1+\alpha)_n (a+b+2)_n} {}_3F_2 \left[\begin{matrix} -n, -\alpha-n, b+1 \\ -a-n, \beta+1 \end{matrix} ; 1 \right] \quad (7.5.4)$$

$$= \frac{(1+b)_n \Gamma(1+\alpha+\beta+2n)}{(1+\alpha)_n \Gamma(1+\alpha+\beta+n)(a+b+2)_n} {}_3F_2 \left[\begin{matrix} -n, -\beta-n, -a-b-n-1; \\ -b-n, -\alpha-\beta-2n \end{matrix} ; 1 \right] \quad (7.5.5)$$

Proof Comparing the integral (7.4.1) with the remaining five integrals from (7.4.2) to (7.4.6), we can obtain the above five hypergeometric transformations associated with terminating Clausen's series ${}_3F_2$.

7.6 Special Cases

It is to be noted that the reciprocal of Gamma of non-positive integers is always zero.

Case (i)

Putting $a = \alpha$, $b = \sigma$ in equations (7.4.1), (7.4.2), (7.4.3), (7.4.4), (7.4.5), applying Gauss theorem (5.1.9) and putting $a = \alpha$, $b = \sigma$ in equation (7.4.6), applying Saalschütz' summation theorem (5.1.10), we get

$$\int_{-1}^{+1} (1-x)^\alpha (1+x)^\sigma P_n^{(\alpha, \beta)}(x) dx = \frac{2^{\alpha+\sigma+1} \Gamma(\sigma+1) \Gamma(\alpha+n+1) \Gamma(\sigma-\beta+1)}{\Gamma(\sigma-\beta-n+1) \Gamma(\alpha+\sigma+n+2) n!} \quad (7.6.1)$$

$$(Re(\alpha) > -1, Re(\sigma) > -1)$$

which is the correct form of the integral (7.1.13).

Case (ii)

Putting $a = \rho$, $b = \beta$ in equations (7.4.1), (7.4.2), (7.4.3), (7.4.4), (7.4.5), applying Saalschütz' summation theorem (5.1.10) and putting $a = \rho$, $b = \beta$ in equation (7.4.6), applying Gauss theorem (5.1.9), we get

$$\int_{-1}^{+1} (1-x)^\rho (1+x)^\beta P_n^{(\alpha, \beta)}(x) dx = \frac{2^{\rho+\beta+1} \Gamma(\rho+1) \Gamma(\beta+n+1) \Gamma(\alpha-\rho+n)}{n! \Gamma(\alpha-\rho) \Gamma(\rho+\beta+n+2)} \quad (7.6.2)$$

$$(Re(\rho) > -1, Re(\beta) > -1)$$

which is the known integral [26, p.284 (16.4.2)].

Case (iii)

Putting $a = \alpha$ and $b = \beta$ in equations (7.4.1) and (7.4.4), applying Gauss theorem (5.1.9), we get

$$\int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) dx = \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+n+1) \Gamma(\beta+1)}{(n)! \Gamma(\alpha+\beta+n+2) \Gamma(1-n)} \quad (7.6.3)$$

$$= \begin{cases} 0, & \text{when } n = 1, 2, 3, \dots \\ \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{when } n = 0 \end{cases}$$

where $Re(\alpha) > -1$, $Re(\beta) > -1$.

Case (iv)

Putting $a = \alpha$, $b = \beta$ in equation (7.4.2) and using combinatorial identity (7.1.15), we get

$$\int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) dx = \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{(n)! \Gamma(\alpha+\beta+n+2)} \sum_{k=0}^n \frac{(-n)_k}{k!} \quad (7.6.4)$$

$$= \begin{cases} 0, & \text{when } n = 1, 2, 3, \dots \\ \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{when } n = 0 \end{cases}$$

where $Re(\alpha) > -1$, $Re(\beta) > -1$.

Case (v)

Putting $a = \rho$ and $b = \sigma$ in equation (7.4.1), we get

$$\begin{aligned} & \int_{-1}^{+1} (1-x)^\rho (1+x)^\sigma P_n^{(\alpha, \beta)}(x) dx \\ &= \frac{2^{\rho+\sigma+1} \Gamma(\rho+1) \Gamma(\sigma+1)}{\Gamma(\rho+\sigma+2)} \frac{(1+\alpha)_n}{n!} {}_3F_2 \left[\begin{matrix} -n, \alpha+\beta+n+1, \rho+1; \\ 1+\alpha, \rho+\sigma+2; \end{matrix} \middle| 1 \right] \end{aligned} \quad (7.6.5)$$

$$(Re(\rho) > -1 \quad , \quad Re(\sigma) > -1)$$

which is the correct form of the integral (7.1.14).

Case (vi)

Putting $a = \alpha$, $b = \beta$ in equations (7.4.3), (7.4.6) and applying Gauss theorem (5.1.9), we get

$$\int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) dx = \frac{(-1)^n 2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+n+1)}{(n)! \Gamma(\alpha+\beta+n+2) \Gamma(1-n)} \quad (7.6.6)$$

$$= \begin{cases} 0, & \text{when } n = 1, 2, 3, \dots \\ \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{when } n = 0 \end{cases}$$

where $Re(\alpha) > -1$, $Re(\beta) > -1$.

The integral (7.6.6) can be obtained from the integral (7.6.3) if we interchange α and β in equation (7.6.3), use the property of definite integral (7.1.17) and a result (7.1.16) on Jacobi Polynomials.

Case (vii)

Putting $a = \alpha$, $b = \beta$ in equation (7.4.5) and using combinatorial identity (7.1.15), we get

$$\begin{aligned} & \int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) dx \\ &= \frac{(-1)^n 2^{(\alpha+\beta+1)} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{(n)! \Gamma(\alpha+\beta+n+2)} \sum_{k=0}^n \frac{(-n)_k}{k!} \end{aligned} \quad (7.6.7)$$

$$= \begin{cases} 0, & \text{when } n = 1, 2, 3, \dots \\ \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{when } n = 0 \end{cases}$$

where $Re(\alpha) > -1$, $Re(\beta) > -1$.

The integral (7.6.7) can also be obtained from the integral (7.6.4) using the same technique as discussed in case (vi).

Chapter 8

Srivastava-Buschman Type Generating Relations Using Series Rearrangement Techniques

8.1 Introduction

Motivated by several demonstrated applications of results on generating functions for the generalized hypergeometric ${}_pF_q$ function, with the help of series arrangement technique, we have presented here some results which are unification and generalizations in character and also include as particular cases a number of well known results given by Srivastava, Buschman, Burchnall, Chaundy, Shively and Krall & Frink. Along with these, some interesting applications of these general results are also investigated.

Two important closed forms [85, p.70, Q.10]

$$(1-z)^{-\frac{1}{2}} \left[\frac{2}{1+\sqrt{(1-z)}} \right]^{(2\gamma-1)} = {}_2F_1 \left[\begin{matrix} \gamma, \gamma + \frac{1}{2}; \\ 2\gamma \end{matrix} ; z \right] \quad (8.1.1)$$

Further,

$$\left[\frac{2}{1+\sqrt{(1-z)}} \right]^{(2\gamma-1)} = {}_2F_1 \left[\begin{matrix} \gamma, \gamma - \frac{1}{2}; \\ 2\gamma \end{matrix} ; z \right] \quad (8.1.2)$$

Properties of Pochhammer's Symbol

$$(a)_{r-p} = (a)_r (a+r)_{-p} \quad (8.1.3)$$

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_n}; a \neq 0, \pm 1, \pm 2, \dots \quad (8.1.4)$$

$$(n-mk)! = \frac{(-1)^{mk} n!}{(-n)_{mk}} \quad (8.1.5)$$

$$\frac{(-1)^p (\lambda)_{2p}}{(\lambda)_p p!} = \binom{-\lambda-p}{p} \quad (8.1.6)$$

Gauss's multiplication formula

$$(\lambda)_{kn} = k^{kn} \left(\frac{\lambda}{k} \right)_n \left(\frac{\lambda+1}{k} \right)_n \cdots \left(\frac{\lambda+k-1}{k} \right)_n \quad (8.1.7)$$

where $n = 0, 1, 2, 3, \dots$ and k is positive integer.

Summation properties [117, p.101 Lemma 3 (5)]

For any positive integer 'm'

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{m} \right]} A(k, n - mk) \quad (8.1.8)$$

where, and in what follows, $[x]$ denotes the greatest integer in x and the series involved are absolutely convergent.

$$\sum_{s=0}^n \Phi(s) = \sum_{s=0}^n \Phi(n-s) \quad (8.1.9)$$

$${}_1F_1 \left[\begin{matrix} -n; \\ 1+\beta; \end{matrix} z \right] = \frac{(-z)^n}{(1+\beta)_n} {}_2F_0 \left[\begin{matrix} -n, -\beta-n; \\ \text{---} \end{matrix} ; \frac{-1}{z} \right] \quad (8.1.10)$$

Rainville Polynomial [85, p.294, eq.(3)]

$$\varphi_n(c, x) = \sum_{k=0}^n \frac{(-1)^k (c)_{n+k} x^k}{k! (n-k)!} = \frac{(c)_n}{n!} {}_2F_0 \left[\begin{matrix} -n, c+n; \\ \text{---} \end{matrix} ; x \right] \quad (8.1.11)$$

Simple Bessel Polynomial [85, p.293, 150(1)]

$$y_n(x, 2, 2) = y_n(x) = {}_2F_0 \left[\begin{matrix} -n, 1+n; \\ \text{---} \end{matrix} ; \frac{-x}{2} \right] \quad (8.1.12)$$

Krall and Frink's generalized Bessel polynomial [117, p.75, (1.9.1);

see also 85, p.294, 150(2) and 48]

$$y_n(x, \alpha, \beta) = {}_2F_0 \left[\begin{matrix} -n, \alpha+n-1; \\ \text{---} \end{matrix} ; \frac{-x}{\beta} \right] \quad (8.1.13)$$

Shively's Pseudo-Laguerre Polynomials [85, p.298, eq.(1) and (3)]

$$R_n(a, x) = \frac{(a)_{2n}}{n! (a)_n} {}_1F_1 \left[\begin{matrix} -n; \\ a+n; \end{matrix} x \right] = \frac{1}{(a-1)_n} \sum_{k=0}^n \frac{(a-1)_{n+k} L_{n-k}(x)}{k!} \quad (8.1.14)$$

where $L_n(x)$ is called Simple Laguerre polynomial.

Shively's Polynomial [96, p.54; 85, p.299, eq.(6)]

$$S_n^\lambda(x) = \frac{(\lambda)_{2n}}{(\lambda)_n n!} {}_{p+1}F_{q+1} \left[\begin{matrix} -n, \alpha_1, \dots, \alpha_p; \\ \lambda+n, \beta_1, \dots, \beta_q; \end{matrix} x \right] \quad (8.1.15)$$

Burchall's Generating Relation [85, p.296, (15); 84, p.104, (4); see also 9]

$$(1+4xt)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{(1+4xt)}} \right)^{(c-1)} \exp \left(\frac{2t}{1+\sqrt{(1+4xt)}} \right) = \sum_{n=0}^{\infty} \frac{\varphi_n(c, x) t^n}{(c)_n}$$

Toscano's Generating Relation [85, p.298, eq. (3)]

$$(1-4t)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{(1-4t)}} \right)^{a-1} \exp \left(\frac{-4xt}{(1+\sqrt{(1-4t)})^2} \right) = \sum_{n=0}^{\infty} R_n(a, x) t^n \quad (8.1.16)$$

8.2 First Set of Generating Relations (8.2.1) and (8.2.6)

If M and N are positive integers and other parameters and variables are adjusted in such a way that each term is well defined, then

$$\begin{aligned}
 & (1-Dyt^{Nq})^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1-Dyt^{Nq})}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1-Dyt^{Nq})}} \right\}^M \right] \\
 &= \sum_{r=0}^{\infty} \frac{(a_1)_r \dots (a_A)_r (Gx)^r t^{qr}}{(b_1)_r \dots (b_B)_r r!} {}_{NB+N+MN-1} F_{NA+MN-2} \left[\begin{matrix} \frac{1-(b_B)-r}{N}, \frac{2-(b_B)-r}{N}, \dots, \frac{N-(b_B)-r}{N}, \\ \frac{1-(a_A)-r}{N}, \frac{2-(a_A)-r}{N}, \dots, \frac{N-(a_A)-r}{N}, \\ \Delta(N; -r), \Delta(MN-1; 1-Mr-\lambda); \\ \Delta(MN-2; 1-Mr-\lambda); \end{matrix} \right. \\
 & \quad \left. \frac{(-N)^{N+N(B-A)}(1-MN)^{(MN-1)} (Dy)}{(2-MN)^{(MN-2)} (Gx)^N 4} \right] \quad (8.2.1)
 \end{aligned}$$

Derivation:

Suppose left hand side of equation (8.2.1) is denoted by Υ , then expanding ${}_A F_B$ into series form and using closed form (8.1.1), again expanding ${}_2 F_1$ into series form, we get

$$\Upsilon = \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a_1)_r \dots (a_A)_r (Gx)^r}{(b_1)_r \dots (b_B)_r r!} \frac{\left(\frac{Mr+\lambda}{2}\right)_p}{p!(Mr+\lambda)_p} \frac{\left(\frac{Mr+\lambda+1}{2}\right)_p}{p!} (Dy)^p t^{Npq+qr} \quad (8.2.2)$$

Replacing 'r' by 'r-Np' by using summation property, we get

$$\Upsilon = \sum_{r=0}^{\infty} \sum_{p=0}^{\left[\frac{r}{N}\right]} \frac{(a_1)_{r-Np} \dots (a_A)_{r-Np} (Gx)^{r-Np}}{(b_1)_{r-Np} \dots (b_B)_{r-Np} (r-Np)!} \frac{\left(\frac{M(r-Np)+\lambda}{2}\right)_p}{p!(M(r-Np)+\lambda)_p} \frac{\left(\frac{M(r-Np)+\lambda+1}{2}\right)_p}{p!} (Dy)^p t^{qr} \quad (8.2.3)$$

$$\begin{aligned}
 &= \sum_{r=0}^{\infty} \sum_{p=0}^{\left[\frac{r}{N}\right]} \frac{(a_1)_r (a_1+r)_{-Np} \dots (a_A)_r (a_A+r)_{-Np} (Gx)^{r-Np}}{(b_1)_r (b_1+r)_{-Np} \dots (b_B)_r (b_B+r)_{-Np} (r-Np)!} \\
 & \quad \cdot \frac{(Mr-MNp+\lambda)_{2p}}{p! 2^{2p} (M(r-Np)+\lambda)_p} (Dy)^p t^{qr} \quad (8.2.4)
 \end{aligned}$$

After further simplification, we get

$$\Upsilon = \sum_{r=0}^{\infty} \sum_{p=0}^{\left[\frac{r}{N}\right]} \frac{(a_1)_r \dots (a_A)_r (-1)^{\{Np(A-B+1)-p\}} (1-b_1-r)_{Np} \dots (1-b_B-r)_{Np}}{(b_1)_r \dots (b_B)_r (1-a_1-r)_{Np} \dots (1-a_A-r)_{Np}} \cdot \frac{(-r)_{Np} (Gx)^{r-Np}}{r!} \frac{(1-Mr-\lambda)_{(MN-1)p}}{p! 2^{2p} (1-Mr-\lambda)_{(MN-2)p}} (Dy)^p t^{qr} \quad (8.2.5)$$

Using multiplication theorem and after simplification we get the required R.H.S. of equation (8.2.1).

Special case

Taking $q = 1, G = 1$ and $x = 1$ in the main result (8.2.1), we obtain

$$\begin{aligned} & (1 - Dy t^N)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dy t^N)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} t \left\{ \frac{2}{1 + \sqrt{(1 - Dy t^N)}} \right\}^M \right] \\ &= \sum_{r=0}^{\infty} \frac{(a_1)_r \dots (a_A)_r t^r}{(b_1)_r \dots (b_B)_r r!} {}_{NB+N+MN-1} F_{NA+MN-2} \left[\begin{matrix} \frac{1-(b_B)-r}{N}, \frac{2-(b_B)-r}{N}, \dots, \frac{N-(b_B)-r}{N}, \\ \frac{1-(a_A)-r}{N}, \frac{2-(a_A)-r}{N}, \dots, \frac{N-(a_A)-r}{N}, \\ \Delta(N; -r), \Delta(MN-1; 1-Mr-\lambda); \\ \Delta(MN-2; 1-Mr-\lambda); \end{matrix} \frac{(-N)^{N+N(B-A)} (1-MN)^{(MN-1)} (Dy)}{(2-MN)^{(MN-2)} 4} \right] \quad (8.2.6) \end{aligned}$$

8.3 Second Set of Generating Relations (8.3.1), (8.3.2), (8.3.3), (8.3.4), (8.3.5) and (8.3.6)

Let M and N are positive integers and other parameters and variables are adjusted in such a way that each term is well defined. Then

Case(i). When $q > M \Rightarrow 2q > M$

$$\left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^M \right]$$

$$\begin{aligned}
&= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p}}{(\lambda)_p} \frac{(Dyt)^p}{p! 4^p} {}_{A+2q-M+1}F_{B+2q-M+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+M-1}{M}, \\ b_1, \dots, b_B, \Delta(2q-M; 2-\lambda-2p), \\ \Delta(q-M; 1-\lambda-p); \\ \frac{\lambda-1}{M} \end{matrix} \right. \\
&\quad \left. \frac{(4q)^q (q-M)^{(q-M)} Gx}{(Dy)^q (2q-M)^{(2q-M)}} \right] \quad (8.3.1)
\end{aligned}$$

Case(ii). When $q < M$ and $M - 2q > 0$

$$\begin{aligned}
&\left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^M \right] \\
&= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p}}{(\lambda)_p} \frac{(Dyt)^p}{p! 4^p} {}_{A+M-q+1}F_{B+M-q+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+M-1}{M}, \\ b_1, \dots, b_B, \Delta(M-q; \lambda+p), \\ \Delta(M-2q; \lambda-1+2p); \\ \frac{\lambda-1}{M} \end{matrix} \right. \\
&\quad \left. \frac{(-4q)^q (M-2q)^{(M-2q)} Gx}{(Dy)^q (M-q)^{(M-q)}} \right] \quad (8.3.2)
\end{aligned}$$

Case(iii). When $q < M$ and $M - 2q < 0$

$$\begin{aligned}
&\left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^M \right] \\
&= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p}}{(\lambda)_p} \frac{(Dyt)^p}{p! 4^p} {}_{A+q+1}F_{B+q+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+M-1}{M} \\ b_1, \dots, b_B, \Delta(M-q; \lambda+p), \\ \frac{\lambda-1}{M}, \Delta(2q-M; 2-\lambda-2p); \\ \frac{(-4q)^q (-1)^M Gx}{(Dy)^q (M-q)^{(M-q)} (2q-M)^{(2q-M)}} \end{matrix} \right] \quad (8.3.3)
\end{aligned}$$

Case(iv). When $q < M$ and $M - 2q = 0$

$$\left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{2q} \right]$$

$$= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p} (Dyt)^p}{(\lambda)_p p! 4^p} {}_{A+q+1}F_{B+q+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+2q-1}{2q}; \frac{(-4)^q Gx}{(Dy)^q} \\ b_1, \dots, b_B, \frac{\lambda-1}{2q}, \Delta(q; \lambda+p); \end{matrix} \right] \quad (8.3.4)$$

Case(v). When $q = M$ and $2q - M > 0$

$$\begin{aligned} & \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^q \right] \\ &= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p} (Dyt)^p}{(\lambda)_p p! 4^p} {}_{A+q+1}F_{B+q+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+q-1}{q}; \frac{(4)^q Gx}{(Dy)^q} \\ b_1, \dots, b_B, \frac{\lambda-1}{q}, \Delta(q; 2 - \lambda - 2p); \end{matrix} \right] \end{aligned} \quad (8.3.5)$$

Case (vi). When $q = 1$

$$\begin{aligned} & \left(\frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt \left(\frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right)^M \right] \\ &= \sum_{r=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_r (Gxt)^r}{\prod_{j=1}^B (b_j)_r r!} {}_{NB+2MN+N-1}F_{NA+2MN-2} \left[\begin{matrix} \frac{1-(b_B)-r}{N}, \dots, \frac{N-(b_B)-r}{N}, \Delta(N; -r), \\ \frac{1-(a_A)-r}{N}, \dots, \frac{N-(a_A)-r}{N}, \\ \Delta(MN; 2 - Mr - \lambda), \Delta(MN - 1; 1 - Mr - \lambda); \\ \Delta(MN - 2; 2 - Mr - \lambda), \Delta(MN; 1 - Mr - \lambda); \end{matrix} \right. \\ & \quad \left. \frac{(-N)^{(B-A+1)N} (1-MN)^{(MN-1)} Dy}{4(2-MN)^{(MN-2)} (Gx)^N} \right] \end{aligned} \quad (8.3.6)$$

Derivation- We consider the expression:

$$\Theta^* = \left(\frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right)^M \right]$$

Expanding ${}_A F_B$ into series form, using closed form (8.1.2) of ${}_2F_1$, then again expanding ${}_2F_1$ into series form, we get

$$\Theta^* = \sum_{r,p=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_r (Gx)^r}{\prod_{j=1}^B (b_j)_r r!} \frac{\left(\frac{Mr+\lambda}{2}\right)_p \left(\frac{Mr+\lambda-1}{2}\right)_p (Dy)^p}{(Mr+\lambda)_p p!} t^{Np+qr} \quad (8.3.7)$$

For evaluating further, we need to impose following two conditions i.e. $N = 1$ and $q = 1$ which are further classified into several cases. These cases are mentioned below.

Condition I:- When $N = 1$, equation (8.3.7) reduces to

$$\begin{aligned} & \left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} G x t^q \left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^M \right] \\ &= \sum_{r,p=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} \frac{\left(\frac{Mr+\lambda}{2} \right)_p \left(\frac{Mr+\lambda-1}{2} \right)_p (Dy)^p (Gx)^r}{(Mr + \lambda)_p p! r!} t^{p+qr} \end{aligned} \quad (8.3.8)$$

Replacing p by $p - qr$ by using the summation property (8.1.8) and then Gauss's multiplication theorem (8.1.7) in (8.3.8), we have

$$\begin{aligned} & \left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} G x t^q \left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^M \right] \\ &= \sum_{p=0}^{\infty} \frac{(\lambda - 1)_{2p} (Dy t)^p}{(\lambda)_p 4^p p!} \sum_{r=0}^{[p/q]} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} \frac{(\lambda - 1 + 2p)_{r(M-2q)} (-p)_{qr} (\lambda)_{Mr} (Gx)^r 4^{qr}}{(\lambda + p)_{r(M-q)} (-1)^{qr} (\lambda - 1)_{Mr} r! (Dy)^{qr}} \end{aligned} \quad (8.3.9)$$

Case(i). When $q > M \Rightarrow 2q > M$ in (8.3.9), then equation (8.3.9) can be rewritten as

$$\Theta^* = \sum_{p=0}^{\infty} \frac{(\lambda - 1)_{2p} (Dy t)^p}{(\lambda)_p 4^p p!} \sum_{r=0}^{[p/q]} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} \frac{(\lambda - 1 + 2p)_{-r(2q-M)} (-p)_{qr} (\lambda)_{Mr} (Gx)^r 4^{qr}}{(\lambda + p)_{-r(q-M)} (-1)^{qr} (\lambda - 1)_{Mr} r! (Dy)^{qr}} \quad (8.3.10)$$

After further simplification and summing up the terms into hypergeometric form, we get the following generating relation (8.3.1).

$$\left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} G x t^q \left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^M \right]$$

$$\begin{aligned}
&= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p} (Dyt)^p}{(\lambda)_p p! 4^p} {}_{A+2q-M+1}F_{B+2q-M+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+M-1}{M}, \\ b_1, \dots, b_B, \Delta(2q-M; 2-\lambda-2p), \\ \Delta(q-M; 1-\lambda-p); \\ \frac{(4q)^q (q-M)^{(q-M)} Gx}{(Dy)^q (2q-M)^{(2q-M)}} \end{matrix} \right] \\
&\quad \frac{\lambda-1}{M} \quad ; \quad
\end{aligned}$$

Case(ii). When $q < M$ and $M - 2q > 0$ in (8.3.9), then using multiplication formula (8.1.7) and after simplifying, summing up the remaining terms into hypergeometric form, we obtain the following result (8.3.2).

$$\begin{aligned}
&\left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^M \right] \\
&= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p} (Dyt)^p}{(\lambda)_p p! 4^p} {}_{A+M-q+1}F_{B+M-q+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+M-1}{M}, \\ b_1, \dots, b_B, \Delta(M-q; \lambda+p), \\ \Delta(M-2q; \lambda-1+2p); \\ \frac{(-4q)^q (M-2q)^{(M-2q)} Gx}{(Dy)^q (M-q)^{(M-q)}} \end{matrix} \right] \\
&\quad \frac{\lambda-1}{M} \quad ; \quad
\end{aligned}$$

Case(iii). When $q < M$ and $M - 2q < 0$ in (8.3.9), then equation (8.3.9) can be rewritten as

$$\begin{aligned}
\Theta^* &= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p}}{(\lambda)_p p!} \left(\frac{Dyt}{4} \right)^p \sum_{r=0}^{[p/q]} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} \frac{(\lambda-1+2p)_{-r(2q-M)} (-p)_{qr} (\lambda)_{Mr}}{(\lambda+p)_{r(M-q)} (-1)^{qr} (\lambda-1)_{Mr}} \\
&\quad \cdot \frac{(Gx)^r 4^{qr}}{r! (Dy)^{qr}} \quad (8.3.11)
\end{aligned}$$

After further simplification and summing up the terms into hypergeometric form, we get the following generating relation (8.3.3).

$$\left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^M \right]$$

$$\begin{aligned}
&= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p}}{(\lambda)_p} \frac{(Dyt)^p}{p! 4^p} {}_{A+q+1}F_{B+q+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+M-1}{M} \\ b_1, \dots, b_B, \Delta(M-q; \lambda+p), \\ \frac{(-4q)^q (-1)^M Gx}{(Dy)^q (M-q)^{(M-q)} (2q-M)^{(2q-M)}} \end{matrix} \right] \\
&\quad ; \\
&\quad \frac{\lambda-1}{M}, \Delta(2q-M; 2-\lambda-2p);
\end{aligned}$$

Case(iv). When $q < M$ and $M - 2q = 0$ in (8.3.9) then equation (8.3.9) can be rewritten as

$$\begin{aligned}
&\left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{2q} \right] \\
&= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p}}{(\lambda)_p} \frac{(Dyt)^p}{p!} \left(\frac{Dyt}{4} \right)^p \sum_{r=0}^{[p/q]} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} \frac{(\lambda-1+2p)_0 (-p)_{qr} (\lambda)_{2qr} (Gx)^r 4^{qr}}{(\lambda+p)_{qr} (-1)^{qr} (\lambda-1)_{2qr} r! (Dy)^{qr}}
\end{aligned} \tag{8.3.12}$$

Taking $(\lambda-1+2p)_0 = 1$ and simplifying further, we find the following result (8.3.4):

$$\begin{aligned}
&\left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{2q} \right] \\
&= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p}}{(\lambda)_p} \frac{(Dyt)^p}{p! 4^p} {}_{A+q+1}F_{B+q+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+2q-1}{2q}; \\ b_1, \dots, b_B, \frac{\lambda-1}{2q}, \Delta(q; \lambda+p); \end{matrix} \frac{(-4)^q Gx}{(Dy)^q} \right]
\end{aligned}$$

Case(v). When $q = M$ and $2q - M > 0$ in (8.3.9) then we can rewrite equation (8.3.9) as:

$$\begin{aligned}
&\left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^q \right] \\
&= \sum_{p=0}^{\infty} \frac{(\lambda-1)_{2p}}{(\lambda)_p} \frac{(Dyt)^p}{p!} \left(\frac{Dyt}{4} \right)^p \sum_{r=0}^{[p/q]} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} \frac{(\lambda-1+2p)_{-qr} (-p)_{qr} (\lambda)_{qr} (Gx)^r 4^{qr}}{(\lambda+p)_0 (-1)^{qr} (\lambda-1)_{qr} r! (Dy)^{qr}}
\end{aligned} \tag{8.3.13}$$

Taking $(\lambda + p)_0 = 1$ and solving further, we find the following result (8.3.5):

$$\begin{aligned} & \left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} \begin{matrix} Gxt^q \left(\frac{2}{1 + \sqrt{(1 - Dy t)}} \right)^q \end{matrix} \right] \\ &= \sum_{p=0}^{\infty} \frac{(\lambda - 1)_{2p}}{(\lambda)_p} \frac{(Dy t)^p}{p! 4^p} {}_{A+q+1} F_{B+q+1} \left[\begin{matrix} a_1, \dots, a_A, \Delta(q; -p), \frac{\lambda+q-1}{q} & ; & (4)^q Gx \\ b_1, \dots, b_B, \frac{\lambda-1}{q}, \Delta(q; 2 - \lambda - 2p); & & (Dy)^q \end{matrix} \right] \end{aligned}$$

Condition II:- When $q = 1$ in (8.3.7)

$$\begin{aligned} \Omega &= \left(\frac{2}{1 + \sqrt{(1 - Dy t^N)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} \begin{matrix} Gxt \left(\frac{2}{1 + \sqrt{(1 - Dy t^N)}} \right)^M \end{matrix} \right] \\ &= \sum_{r,p=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} \frac{\left(\frac{Mr+\lambda}{2} \right)_p \left(\frac{Mr+\lambda-1}{2} \right)_p (Dy)^p (Gx)^r}{(Mr + \lambda)_p p!} \frac{t^{r+Np}}{r!} \end{aligned} \quad (8.3.14)$$

Replacing r by $r - Np$ in (8.3.14) by using the summation property, we get

$$\begin{aligned} & \left(\frac{2}{1 + \sqrt{(1 - Dy t^N)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} \begin{matrix} Gxt \left(\frac{2}{1 + \sqrt{(1 - Dy t^N)}} \right)^M \end{matrix} \right] \\ &= \sum_{r=0}^{\infty} \sum_{p=0}^{\left[\frac{r}{N} \right]} \frac{(a_1)_{r-Np} \dots (a_A)_{r-Np}}{(b_1)_{r-Np} \dots (b_B)_{r-Np}} \frac{(Mr - MNp + \lambda - 1)_{2p} (Dy)^p (Gx)^{r-Np} t^r}{(Mr - MNp + \lambda)_p 4^p (r - Np)! p!} \end{aligned} \quad (8.3.15)$$

using the property of Pochhammers symbol, Gauss's multiplication formula and after simplification summing up the terms into hypergeometric form, we obtain the following result (8.3.6):

$$\begin{aligned} & \left(\frac{2}{1 + \sqrt{(1 - Dy t^N)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} \begin{matrix} Gxt \left(\frac{2}{1 + \sqrt{(1 - Dy t^N)}} \right)^M \end{matrix} \right] \\ &= \sum_{r=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} \frac{(Gxt)^r}{r!} {}_{(NB+2MN+N-1)} F_{(NA+2MN-2)} \left[\begin{matrix} \frac{1-(b_B)-r}{N}, \dots, \frac{N-(b_B)-r}{N}, \Delta(N; -r), \\ \frac{1-(a_A)-r}{N}, \dots, \frac{N-(a_A)-r}{N}, \end{matrix} \right] \end{aligned}$$

$$\left[\begin{array}{l} \Delta(MN; 2 - Mr - \lambda), \Delta(MN - 1; 1 - Mr - \lambda); \\ \Delta(MN - 2; 2 - Mr - \lambda), \Delta(MN; 1 - Mr - \lambda) ; \end{array} \frac{(-N)^{(B-A+1)N} (1-MN)^{(MN-1)} Dy}{4(2-MN)^{(MN-2)} (Gx)^N} \right]$$

8.4 Third Set of Generating Relations (8.4.1),(8.4.3), (8.4.4),(8.4.6),(8.4.8), (8.4.13) and Their Particular Cases

Case (i). When $q > M$

$$\begin{aligned} & (1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{array}{c} (a_A); \\ (b_B); \end{array} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^M \right] \\ &= \sum_{p=0}^{\infty} \binom{-\lambda - p}{p} \left(\frac{-Dyt}{4} \right)^p \cdot {}_{A+2q-M} F_{B+2q-M} \left[\begin{array}{c} (a_A), \Delta(q; -p), \Delta(q - M; 1 - \lambda - p); \\ (b_B), \Delta(2q - M; 1 - \lambda - 2p) \end{array} ; \frac{(4q)^q (q - M)^{(q-M)} (Gx)}{(Dy)^q (2q - M)^{(2q-M)}} \right] \end{aligned} \quad (8.4.1)$$

Particular Case of (8.4.1)

Taking $\lambda = -\alpha$, $A = r$, $B = s$, $D = -4$, $y = 1$, $G = (-1)^q$, $M = q - p$ in equation (8.4.1),

$$\begin{aligned} & (1 + 4t)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 + 4t)}} \right)^{-\alpha-1} {}_r F_s \left[\begin{array}{c} (a_r); \\ (b_s); \end{array} x(-t)^q \left\{ \frac{2}{1 + \sqrt{(1 + 4t)}} \right\}^{q-p} \right] \\ &= \sum_{n=0}^{\infty} \binom{\alpha - n}{n} {}_{r+p+q} F_{s+p+q} \left[\begin{array}{c} (a_r), \Delta(q; -n), \Delta(p; 1 + \alpha - n); \\ (b_s), \Delta(p + q; 1 + \alpha - 2n) \end{array} ; \frac{q^q p^p x}{(p + q)^{(p+q)}} \right] t^n \end{aligned} \quad (8.4.2)$$

which is result of Srivastava-Buschman [117, p.394, Q.22 (iii); see also 111, p.365; 4, p.44].

Case (ii). When $q < M$ and $(2q - M) < 0$

$$(1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{array}{c} (a_A); \\ (b_B); \end{array} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^M \right]$$

$$\begin{aligned}
&= \sum_{p=0}^{\infty} \binom{-\lambda-p}{p} \left(\frac{-Dyt}{4} \right)^p \\
&\cdot {}_{A-q+M}F_{B-q+M} \left[\begin{matrix} (a_A), \Delta(q; -p), \Delta(M-2q; \lambda+2p); \\ (b_B), \Delta(M-q; \lambda+p) \end{matrix} ; \frac{(4q)^q (M-2q)^{(M-2q)}(Gx)}{(-Dy)^q (M-q)^{(M-q)}} \right] \quad (8.4.3)
\end{aligned}$$

Case (iii). When $q < M$ and $(2q - M) > 0$

$$\begin{aligned}
&(1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^M \right] \\
&= \sum_{p=0}^{\infty} \binom{-\lambda-p}{p} {}_{A+q} F_{B+q} \left[\begin{matrix} (a_A), \Delta(q; -p) \\ (b_B), \Delta(2q - M; 1 - \lambda - 2p), \Delta(M - q; \lambda + p); \end{matrix} ; \right. \\
&\quad \left. \frac{(4q)^q (Gx)}{(Dy)^q (2q - M)^{(2q-M)} (q - M)^{(M-q)}} \right] \left(\frac{-Dyt}{4} \right)^p \quad (8.4.4)
\end{aligned}$$

Particular case of (8.4.4):-

Putting $D = 4, y = 1, G = (-1)^q, M = 2q - p + 1, \lambda = \alpha + 1, A = r, B = s$ in (8.4.4), we get a known result of Srivastava-Buschman [117, p.372, 7.4(12); see also 111, p.363, eq. (15)]

$$\begin{aligned}
&(1 - 4t)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - 4t)}} \right)^{\alpha} {}_r F_s \left[\begin{matrix} (a_r); \\ (b_s); \end{matrix} x(-t)^q \left\{ \frac{2}{1 + \sqrt{(1 - 4t)}} \right\}^{2q-p+1} \right] \\
&= \sum_{n=0}^{\infty} \binom{\alpha + 2n}{n} t^n {}_{r+q} F_{s+q} \left[\begin{matrix} (a_r), \Delta(q; -n) \\ (b_s), \Delta(p-1; -\alpha - 2n), \Delta(q-p+1; \alpha + n + 1); \end{matrix} ; \right. \\
&\quad \left. \frac{q^q x}{(1-p)^{(p-1)}(q-p+1)^{(q-p+1)}} \right] \quad (8.4.5)
\end{aligned}$$

provided $p \leq q + 1$.

Case (iv). Taking $q < M$ and $2q = M$

$$(1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^{2q} \right]$$

$$= \sum_{i=0}^{\infty} \begin{pmatrix} -\lambda - i \\ i \end{pmatrix}_{A+q} F_{B+q} \left[\begin{matrix} (a_A), \Delta(q; -i) & ; & \frac{(-4)^q (Gx)}{(Dy)^q} \\ (b_B), \Delta(q; \lambda + i); & & \end{matrix} \right] \left(\frac{-Dyt}{4} \right)^i \quad (8.4.6)$$

Particular case of (8.4.6):-

Putting $D = 4, y = 1, G = -1, q = 1, M = 2$ in (8.4.6) we find a known result:
[96, p.54 – 55; 85, p.298, eq.(5); 117, p.187, Q.49]

$$\begin{aligned} & (1 - 4t)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - 4t)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); & (-xt) \left\{ \frac{2}{1 + \sqrt{(1 - 4t)}} \right\}^2 \\ (b_B); & \end{matrix} \right] \\ &= \sum_{i=0}^{\infty} \frac{t^i (\lambda)_{2i}}{i! (\lambda)_i} {}_{A+1} F_{B+1} \left[\begin{matrix} (a_A), -i & ; \\ (b_B), \lambda + i; & x \end{matrix} \right] = \sum_{n=0}^{\infty} S_n^{(\lambda)}(x) t^n \end{aligned} \quad (8.4.7)$$

Case (v). When $q = M$

$$\begin{aligned} & (1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); & Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^q \\ (b_B); & \end{matrix} \right] \\ &= \sum_{p=0}^{\infty} \begin{pmatrix} -\lambda - p \\ p \end{pmatrix}_{A+q} F_{B+q} \left[\begin{matrix} (a_A), \Delta(q; -p) & ; & \frac{(4)^q (Gx)}{(Dy)^q} \\ (b_B), \Delta(q; 1 - \lambda - 2p); & & \end{matrix} \right] \left(\frac{-Dyt}{4} \right)^p \end{aligned} \quad (8.4.8)$$

Particular Cases of (8.4.8):- [117, p.141, 2.6 (20)]

Putting $D = 4, G = x = 1, A = l, B = m, a = c, b = d, q = 1, M = 1$ and replacing y by x in (8.4.8), we get

$$\begin{aligned} & (1 - 4xt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - 4xt)}} \right)^{\lambda-1} {}_l F_m \left[\begin{matrix} (c_l); & \frac{2t}{1 + \sqrt{(1 - 4xt)}} \\ (d_m); & \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \begin{pmatrix} -\lambda - n \\ n \end{pmatrix}_{l+1} F_{m+1} \left[\begin{matrix} (c_l), -n & ; & x^{-1} \\ (d_m), 1 - \lambda - 2n; & & \end{matrix} \right] (-xt)^n \end{aligned} \quad (8.4.9)$$

Reversing the order of the terms in the hypergeometric polynomial ${}_{l+1} F_{m+1}$, we get Chaundy's generating relation [84, p.104 (4); see also 117, p.141, 2.6 (20)]

$$(1 - 4xt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - 4xt)}} \right)^{\lambda-1} {}_l F_m \left[\begin{matrix} (c_l); & \frac{2t}{1 + \sqrt{(1 - 4xt)}} \\ (d_m); & \end{matrix} \right]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^l (c_j)_n}{\prod_{j=1}^m (d_j)_n} {}_{m+2}F_l \left[\begin{matrix} -n, \lambda + n, 1 - (d_m) - n; \\ 1 - (c_l) - n \end{matrix} ; (-1)^{l+m+1} x \right] \frac{t^n}{n!} \quad (8.4.10)$$

which is the generalization of Burchnell's generating relation.

Putting $D = -4, G = x = 1, A = 0, B = 0, q = 1, M = 1, \lambda = c$ and replacing y by x in (8.4.8), we have

$$\begin{aligned} (1 + 4xt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 + 4xt)}} \right)^{c-1} \exp \left(\frac{2t}{1 + \sqrt{(1 + 4xt)}} \right) \\ = \sum_{r=0}^{\infty} {}_1F_1 \left[\begin{matrix} -r & ; & -1 \\ 1 - c - 2r & ; & x \end{matrix} \right] \cdot \frac{(c)_{2r}}{(c)_r} \frac{(-x t)^r}{r!} \\ = \sum_{n=0}^{\infty} {}_2F_0 \left[\begin{matrix} -n, c + n; \\ \text{---} & ; & x \end{matrix} \right] \frac{t^n}{n!} \end{aligned} \quad (8.4.11)$$

In equation (8.4.11), replacing x by $\frac{-x}{\beta}$, c by $\alpha - 1$ and using the definition of generalized Bessel's polynomial of Krall and Frink, we get the corrected form of the result of Burchnell [9, p.67; see also 117, p.84, eq.(19); 84, p.104 (4)]:

$$\begin{aligned} \left(1 - \frac{4xt}{\beta} \right)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - \frac{4xt}{\beta})}} \right)^{\alpha-2} \exp \left(\frac{2t}{1 + \sqrt{(1 - \frac{4xt}{\beta})}} \right) \\ = \sum_{n=0}^{\infty} {}_2F_0 \left[\begin{matrix} -n, \alpha + n - 1; \\ \text{---} & ; & \frac{-x}{\beta} \end{matrix} \right] \frac{t^n}{n!} \\ = \sum_{n=0}^{\infty} y_n(x, \alpha, \beta) \frac{t^n}{n!} \end{aligned} \quad (8.4.12)$$

Case (vi). When $q = 1$, then

$$\begin{aligned} (1 - Dyt^N)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt \left\{ \frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right\}^M \right] \\ = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} (N + NB + MN - 1) F_{(NA + MN - 2)} \left[\begin{matrix} \Delta(N; -r), \Delta(MN - 1; 1 - Mr - \lambda), \\ \Delta(MN - 2; 1 - Mr - \lambda), \end{matrix} \right] \end{aligned}$$

$$\left[\begin{array}{c} \frac{1-(b_B)-r}{N}, \dots, \frac{N-(b_B)-r}{N}; \\ \frac{1-(a_A)-r}{N}, \dots, \frac{N-(a_A)-r}{N}; \\ \frac{(-N)^{(B-A+1)N}(1-MN)^{(MN-1)}}{(2-MN)^{(MN-2)}(Gx)^N} \frac{Dy}{4} \end{array} \right] \frac{(Gxt)^r}{r!} \quad (8.4.13)$$

Particular case of (8.4.13):-

Taking $G = 1, x = 1, q = 1, y \rightarrow x$ in equation (8.4.13), we get

$$\begin{aligned} & (1 - Dxt^N)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dxt^N)}} \right)^{\lambda-1} {}_A F_B \left[\begin{array}{c} (a_A); \\ (b_B); \end{array} t \left\{ \frac{2}{1 + \sqrt{(1 - Dxt^N)}} \right\}^M \right] \\ &= \sum_{r=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_r}{\prod_{j=1}^B (b_j)_r} \frac{(N+NB+MN-1)F_{(NA+MN-2)}}{(N+NB+MN-1)F_{(NA+MN-2)}} \left[\begin{array}{c} \Delta(N; -r), \Delta(MN-1; 1-Mr-\lambda), \\ \Delta(MN-2; 1-Mr-\lambda), \end{array} \right] \\ & \left[\begin{array}{c} \frac{1-(b_B)-r}{N}, \dots, \frac{N-(b_B)-r}{N}; \\ \frac{1-(a_A)-r}{N}, \dots, \frac{N-(a_A)-r}{N}; \\ \frac{(-N)^{(B-A+1)N}(1-MN)^{(MN-1)}}{(2-MN)^{(MN-2)}(Dx)^N} \frac{(Dx)}{4} \end{array} \right] \frac{(t)^r}{r!} \quad (8.4.14) \end{aligned}$$

Demonstration:- Consider the expression

$$\begin{aligned} \Theta &= (1 - Dyt^N)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right)^{\lambda-1} \\ & {}_A F_B \left[\begin{array}{c} (a_A); \\ (b_B); \end{array} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right\}^M \right] \quad (8.4.15) \end{aligned}$$

Expanding ${}_A F_B$ into series form, using closed form (8.1.1) of ${}_2F_1$, then again expanding ${}_2F_1$ into series form, we get

$$\Theta = \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a_1)_r \dots (a_A)_r}{(b_1)_r \dots (b_B)_r} \frac{(Gx)^r}{r!} \frac{\left(\frac{Mr+\lambda}{2}\right)_p \left(\frac{Mr+\lambda+1}{2}\right)_p}{p!(Mr+\lambda)_p} (Dy)^p t^{Np+qr} \quad (8.4.16)$$

Using multiplication formula, we have

$$\Theta = \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a_1)_r \dots (a_A)_r}{(b_1)_r \dots (b_B)_r} \frac{(Mr+\lambda)_{2p}}{(Mr+\lambda)_p} \frac{(Gx)^r (Dy)^p t^{Np+qr}}{2^{2p} r! p!} \quad (8.4.17)$$

For evaluating further, we need to impose following two conditions i.e. $N = 1$ and $q = 1$ which are further classified into several cases. These cases are mentioned below.

Condition I: When $N = 1$, equation (8.4.17) reduces to

$$\begin{aligned} & (1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^M \right] \\ &= \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(a_1)_r \dots (a_A)_r}{(b_1)_r \dots (b_B)_r} \frac{(Mr + \lambda)_{2p}}{(Mr + \lambda)_p} \frac{(Gx)^r (Dy)^p t^{p+qr}}{2^{2p} r! p!} \end{aligned} \quad (8.4.18)$$

Replacing p by $p - qr$, we get

$$\begin{aligned} & (1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^M \right] \\ &= \sum_{p=0}^{\infty} \frac{(\lambda)_{2p}}{(\lambda)_p} \frac{(Dyt)^p}{4^p p!} \sum_{r=0}^{\left[\frac{p}{q} \right]} \frac{(a_1)_r \dots (a_A)_r (-p)_{qr} (1 - \lambda - p)_{(q-M)r} 4^{qr} (Gx)^r}{(b_1)_r \dots (b_B)_r (1 - \lambda - 2p)_{(2q-M)r} (Dy)^{qr} r!} \end{aligned} \quad (8.4.19)$$

Now using multiplication formula, we obtain

$$\begin{aligned} & (1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^M \right] \\ &= \sum_{p=0}^{\infty} \frac{(\lambda)_{2p} (Dyt)^p}{(\lambda)_p 4^p p!} \sum_{r=0}^{\left[\frac{p}{q} \right]} \frac{(a_1)_r \dots (a_A)_r \left\{ q^{qr} \left(\frac{-p}{q} \right)_r \dots \left(\frac{-p+q-1}{q} \right)_r \right\} (1 - \lambda - p)_{(q-M)r}}{(b_1)_r \dots (b_B)_r (1 - \lambda - 2p)_{(2q-M)r}} \cdot \frac{4^{qr} (Gx)^r}{(Dy)^{qr} r!} \end{aligned} \quad (8.4.20)$$

Case (i). For $q > M$, in eq. (8.4.20), using multiplication formula and summing into hypergeometric form, we get (8.4.1).

Case (ii). For $q < M$ and $(2q - M) < 0$, equation (8.4.20) can be rewritten as

$$\begin{aligned} & (1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^M \right] \\ &= \sum_{p=0}^{\infty} \frac{(\lambda)_{2p} (Dyt)^p}{(\lambda)_p 4^p p!} \sum_{r=0}^{\left[\frac{p}{q} \right]} \frac{(a_1)_r \dots (a_A)_r \left\{ q^{qr} \left(\frac{-p}{q} \right)_r \dots \left(\frac{-p+q-1}{q} \right)_r \right\} (1 - \lambda - p)_{-(M-q)r}}{(b_1)_r \dots (b_B)_r (1 - \lambda - 2p)_{-(M-2q)r}} \cdot \frac{4^{qr} (Gx)^r}{(Dy)^{qr} r!} \end{aligned} \quad (8.4.21)$$

After simplification, we get (8.4.3).

Case (iii). For $q < M$ and $(2q - M) > 0$, then equation (8.4.20) can be rewritten as

$$\begin{aligned}
& (1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^M \right] \\
&= \sum_{p=0}^{\infty} \frac{(\lambda)_{2p} (Dyt)^p}{(\lambda)_p 4^p p!} \sum_{r=0}^{\left[\frac{p}{q}\right]} \frac{(a_1)_r \dots (a_A)_r \left\{ q^{qr} \left(\frac{-p}{q} \right)_r \dots \left(\frac{-p+q-1}{q} \right)_r \right\} (1 - \lambda - p)_{-(M-q)r}}{(b_1)_r \dots (b_B)_r (1 - \lambda - 2p)_{(2q-M)r}} \\
& \quad \frac{4^{qr} (Gx)^r}{(Dy)^{qr} r!} \tag{8.4.22}
\end{aligned}$$

After simplification, we get (8.4.4).

Case (iv). For $q < M$ and $(2q - M) = 0$, then equation (8.4.20) can be rewritten as

$$\begin{aligned}
& (1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^{2q} \right] \\
&= \sum_{p=0}^{\infty} \frac{(\lambda)_{2p} (Dyt)^p}{(\lambda)_p 4^p p!} \sum_{r=0}^{\left[\frac{p}{q}\right]} \frac{(a_1)_r \dots (a_A)_r \left\{ q^{qr} \left(\frac{-p}{q} \right)_r \dots \left(\frac{-p+q-1}{q} \right)_r \right\} (1 - \lambda - p)_{-(2q-q)r}}{(b_1)_r \dots (b_B)_r (1 - \lambda - 2p)_{(2q-2q)r}} \\
& \quad \frac{4^{qr} (Gx)^r}{(Dy)^{qr} r!} \tag{8.4.23}
\end{aligned}$$

After simplification, we get (8.4.6).

Case (v). For $q = M$, then equation (8.4.20) can be rewritten as

$$\begin{aligned}
& (1 - Dyt)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt^q \left\{ \frac{2}{1 + \sqrt{(1 - Dyt)}} \right\}^q \right] \\
&= \sum_{p=0}^{\infty} \frac{(\lambda)_{2p} (Dyt)^p}{(\lambda)_p 4^p p!} \sum_{r=0}^{\left[\frac{p}{q}\right]} \frac{(a_1)_r \dots (a_A)_r \left\{ q^{qr} \left(\frac{-p}{q} \right)_r \dots \left(\frac{-p+q-1}{q} \right)_r \right\} (1 - \lambda - p)_{(0)r}}{(b_1)_r \dots (b_B)_r (1 - \lambda - 2p)_{(q)r}} \\
& \quad \frac{4^{qr} (Gx)^r}{(Dy)^{qr} r!} \tag{8.4.24}
\end{aligned}$$

After simplifying further, we get (8.4.8).

Condition II: Putting $q = 1$ in (8.4.17), we get

$$\Xi = (1 - Dyt^N)^{-\frac{1}{2}} \left(\frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right)^{\lambda-1} {}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} Gxt \left\{ \frac{2}{1 + \sqrt{(1 - Dyt^N)}} \right\}^M \right]$$

$$= \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{\prod_{j=1}^A (a_j)_r (Mr + \lambda)_{2p}}{B \prod_{j=1}^B (b_j)_r 4^p (Mr + \lambda)_p} \frac{(Gx)^r (Dy)^p t^{Np+r}}{r! p!} \quad (8.4.25)$$

Replacing 'r' by 'r-Np' in (8.4.25), we get

$$\Xi = \sum_{r=0}^{\infty} \sum_{p=0}^{\left[\frac{r}{N}\right]} \frac{(a_1)_{r-Np} \dots (a_A)_{r-Np} (Gx)^{r-Np} (Mr - MNp + \lambda)_{2p}}{(b_1)_{r-Np} \dots (b_B)_{r-Np} (r - Np)! 2^{2p} p! (Mr - MNp + \lambda)_p} (Dy)^p t^r \quad (8.4.26)$$

$$= \sum_{r=0}^{\infty} \sum_{p=0}^{\left[\frac{r}{N}\right]} \frac{(a_1)_r (a_1 + r)_{-Np} \dots (a_A)_r (a_A + r)_{-Np} (Mr + \lambda)_{-(MN-2)p}}{(b_1)_r (b_1 + r)_{-Np} \dots (b_B)_r (b_B + r)_{-Np} (1 + r)_{-Np} (Mr + \lambda)_{-(MN-1)p}} \cdot \frac{(Gx)^{r-Np}}{r! p! 2^{2p}} (Dy)^p t^r \quad (8.4.27)$$

$$= \sum_{r=0}^{\infty} \sum_{p=0}^{\left[\frac{r}{N}\right]} \frac{(a_1)_r \dots (a_A)_r (-1)^{\{Np(A-B-1)-p\}} (1 - b_1 - r)_{Np} \dots (1 - b_B - r)_{Np} (-r)_{Np}}{(b_1)_r \dots (b_B)_r (1 - a_1 - r)_{Np} \dots (1 - a_A - r)_{Np}} \cdot \frac{(1 - Mr - \lambda)_{(MN-1)p} (Gx)^{r-Np}}{(1 - Mr - \lambda)_{(MN-2)p} r! p! 2^{2p}} (Dy)^p t^r \quad (8.4.28)$$

Using multiplication theorem and after simplification, we get (8.4.13).

Chapter 9

**All Possible Generating
Relations Involving Generalized
Polynomials $S_n^{(\alpha,\beta)}(x; m)$,
 $T_n^{(\alpha,\beta)}(x; m, \lambda, \mu)$ And
Hypergeometric Polynomials
 $H_n^{(\alpha,\beta)}(x; m)$, $\mathcal{B}_n^{(\alpha,\beta)}(x; m, \lambda, \mu)$**

9.1 Introduction

This chapter culminates into eighteen generating relations involving generalized polynomials $S_n^{(\alpha, \beta)}(x; m)$ and $T_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ using different combinations of terms. In the form of particular cases, we have also investigated a number of generating relations on Srivastava's generalized hypergeometric polynomials of one variable $H_n^{(\alpha, \beta)}(x; m)$ and Pathan's generalized hypergeometric polynomials of one variable $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$. The results are believed to be new to the author's knowledge and are not found in the literature of generating relations.

The polynomial $S_n^{(\alpha, \beta)}(x; m)$ [117, p.390, Q. 15 (ii)] is defined by

$$S_n^{(\alpha, \beta)}(x; m) = \sum_{r=0}^{\left[\frac{n}{m}\right]} \binom{\alpha + (\beta + 1)n}{n - mr} \gamma_r x^r \quad (9.1.1)$$

and let the polynomial $T_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ be given by

$$T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) = \frac{1}{(\lambda + \mu n)} \sum_{r=0}^{\left[\frac{n}{m}\right]} \left\{ \binom{\alpha + (\beta + 1)n}{n - mr} [\lambda + \mu n + \mu(\beta + 1 - m)r] \right\} \gamma_r x^r \quad (9.1.2)$$

where α, β are complex parameters independent of 'n'; m is an arbitrary positive integer and $\{\gamma_r\}$ is a bounded sequence of arbitrary real and complex numbers such that $\gamma_r \neq 0$.

Srivastava's generalized hypergeometric polynomials of one variable $H_n^{(\alpha, \beta)}(x; m)$ [117, p.360, eq.(7.3.3); see also 109, pp.331 – 332] are given by

$$H_n^{(\alpha, \beta)}(x; m) = \binom{\alpha + (\beta + 1)n}{n}_{p+m} F_{q+m} \left[\begin{matrix} \Delta(m; -n), (a_p) \\ \Delta(m; 1 + \alpha + \beta n), (b_q) \end{matrix} ; x \right] \quad (9.1.3)$$

Pathan's generalized hypergeometric polynomials of one variable $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ are given by

$$\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) = \binom{\alpha + (\beta + 1)n}{n} \cdot {}_{p+m+1}F_{q+m+1} \left[\begin{matrix} \Delta(m; -n), 1 + \frac{\lambda + \mu n}{\mu(\beta + 1 - m)}, (a_p) \\ \Delta(m; 1 + \alpha + \beta n), \frac{\lambda + \mu n}{\mu(\beta + 1 - m)}, (b_q) \end{matrix} ; x \right] \quad (9.1.4)$$

where α, β are complex parameters independent of ' n ' and $\Delta(m; \lambda)$ abbreviates the array of m number of parameters given by

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \dots, \frac{\lambda+m-1}{m}; m \in \mathbb{N}$$

Some results on Pochhammer's symbol after taking all possible combinations:

$$\frac{(\lambda + \mu n) \left(\frac{\lambda + \mu n}{\mu(\beta+1-m)} + 1 \right)_r}{\left(\frac{\lambda + \mu n}{\mu(\beta+1-m)} \right)_r} = \lambda + \mu n + \mu(\beta+1)r - \mu m r \quad (9.1.5)$$

$$\alpha + m(\beta+1)r = \alpha \frac{\left(\frac{\alpha}{m(\beta+1)} + 1 \right)_r}{\left(\frac{\alpha}{m(\beta+1)} \right)_r} \quad (9.1.6)$$

$$\mu m r + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \frac{\mu \zeta (\beta+1) m r}{(1 - \beta \zeta)} = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\zeta \alpha}{m(\zeta+1)} + 1 \right)_r}{\left(\frac{\zeta \alpha}{m(\zeta+1)} \right)_r} \quad (9.1.7)$$

$$\begin{aligned} \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu(\beta+1)r + \frac{\mu \zeta m \beta r}{(1 - \beta \zeta)} + \frac{\mu \zeta m r}{(1 - \beta \zeta)} \\ = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\zeta \alpha}{(\beta+1)(1-\beta \zeta+m \zeta)} + 1 \right)_r}{\left(\frac{\zeta \alpha}{(\beta+1)(1-\beta \zeta+m \zeta)} \right)_r} \end{aligned} \quad (9.1.8)$$

$$\lambda + \mu m r + \frac{\mu \zeta m(\beta+1)r}{(1 - \beta \zeta)} = \lambda \frac{\left(\frac{\lambda(1-\beta \zeta)}{\mu m(\zeta+1)} + 1 \right)_r}{\left(\frac{\lambda(1-\beta \zeta)}{\mu m(\zeta+1)} \right)_r} \quad (9.1.9)$$

$$\lambda + \mu(\beta+1)r + \frac{\mu \zeta m(\beta+1)r}{(1 - \beta \zeta)} = \lambda \frac{\left(\frac{\lambda(1-\beta \zeta)}{\mu(\beta+1)(1-\beta \zeta+m \zeta)} + 1 \right)_r}{\left(\frac{\lambda(1-\beta \zeta)}{\mu(\beta+1)(1-\beta \zeta+m \zeta)} \right)_r} \quad (9.1.10)$$

$$\lambda + \mu m r + \frac{\mu \zeta m \beta r}{(1 - \beta \zeta)} = \lambda \frac{\left(\frac{\lambda(1-\beta \zeta)}{\mu m} + 1 \right)_r}{\left(\frac{\lambda(1-\beta \zeta)}{\mu m} \right)_r} \quad (9.1.11)$$

$$\frac{\mu \zeta (\alpha + m r)}{(1 - \beta \zeta)} = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\alpha}{m} + 1 \right)_r}{\left(\frac{\alpha}{m} \right)_r} \quad (9.1.12)$$

$$\lambda + \mu(\beta+1)r + \frac{\mu \zeta m \beta r}{(1 - \beta \zeta)} = \lambda \frac{\left(\frac{\lambda(1-\beta \zeta)}{\mu[\beta+1+\beta \zeta(m-\beta-1)]} + 1 \right)_r}{\left(\frac{\lambda(1-\beta \zeta)}{\mu[\beta+1+\beta \zeta(m-\beta-1)]} \right)_r} \quad (9.1.13)$$

$$\lambda + \frac{\mu \zeta m(\beta+1)r}{(1 - \beta \zeta)} = \lambda \frac{\left(\frac{\lambda(1-\beta \zeta)}{\mu \zeta m(\beta+1)} + 1 \right)_r}{\left(\frac{\lambda(1-\beta \zeta)}{\mu \zeta m(\beta+1)} \right)_r} \quad (9.1.14)$$

$$\frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu m r = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\zeta \alpha}{m(1 - \beta \zeta)} + 1 \right)_r}{\left(\frac{\zeta \alpha}{m(1 - \beta \zeta)} \right)_r} \quad (9.1.15)$$

$$\frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu(\beta + 1)r = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\zeta \alpha}{(\beta + 1)(1 - \beta \zeta)} + 1 \right)_r}{\left(\frac{\zeta \alpha}{(\beta + 1)(1 - \beta \zeta)} \right)_r} \quad (9.1.16)$$

$$\lambda + \mu m r + \frac{\mu \zeta m r}{(1 - \beta \zeta)} = \lambda \frac{\left(\frac{\lambda(1 - \beta \zeta)}{\mu m(1 - \beta \zeta + \zeta)} + 1 \right)_r}{\left(\frac{\lambda(1 - \beta \zeta)}{\mu m(1 - \beta \zeta + \zeta)} \right)_r} \quad (9.1.17)$$

$$\lambda + \mu(\beta + 1)r + \frac{\mu \zeta m r}{(1 - \beta \zeta)} = \lambda \frac{\left(\frac{\lambda(1 - \beta \zeta)}{\mu[(\beta + 1)(1 - \beta \zeta) + m \zeta]} + 1 \right)_r}{\left(\frac{\lambda(1 - \beta \zeta)}{\mu[(\beta + 1)(1 - \beta \zeta) + m \zeta]} \right)_r} \quad (9.1.18)$$

$$\frac{\mu \zeta(\alpha + \beta m r)}{(1 - \beta \zeta)} = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\alpha}{m \beta} + 1 \right)_r}{\left(\frac{\alpha}{m \beta} \right)_r} \quad (9.1.19)$$

$$\lambda + \frac{\mu \zeta m r}{(1 - \beta \zeta)} = \lambda \frac{\left(\frac{\lambda(1 - \beta \zeta)}{\mu m \zeta} + 1 \right)_r}{\left(\frac{\lambda(1 - \beta \zeta)}{\mu m \zeta} \right)_r} \quad (9.1.20)$$

$$\frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu m r + \frac{\mu \zeta m \beta r}{(1 - \beta \zeta)} = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\zeta \alpha}{m} + 1 \right)_r}{\left(\frac{\zeta \alpha}{m} \right)_r} \quad (9.1.21)$$

$$\frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu(\beta + 1)r + \frac{\mu \zeta m \beta r}{(1 - \beta \zeta)} = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\zeta \alpha}{[(\beta + 1)(1 - \beta \zeta) + m \zeta \beta]} + 1 \right)_r}{\left(\frac{\zeta \alpha}{[(\beta + 1)(1 - \beta \zeta) + m \zeta \beta]} \right)_r} \quad (9.1.22)$$

$$\lambda + \mu m r = \lambda \frac{\left(\frac{\lambda}{m \mu} + 1 \right)_r}{\left(\frac{\lambda}{m \mu} \right)_r} \quad (9.1.23)$$

$$\frac{\lambda + \mu(\beta + 1)r}{\alpha + m(\beta + 1)r} = \frac{\lambda}{\alpha} \frac{\left(\frac{\lambda}{\mu(\beta + 1)} + 1 \right)_r}{\left(\frac{\lambda}{\mu(\beta + 1)} \right)_r} \frac{\left(\frac{\alpha}{m(\beta + 1)} + 1 \right)_r}{\left(\frac{\alpha}{m(\beta + 1)} \right)_r} \quad (9.1.24)$$

$$\lambda + \frac{\mu \zeta \beta m r}{(1 - \beta \zeta)} = \lambda \frac{\left(\frac{\lambda(1 - \beta \zeta)}{\mu m \zeta \beta} + 1 \right)_r}{\left(\frac{\lambda(1 - \beta \zeta)}{\mu m \zeta \beta} \right)_r} \quad (9.1.25)$$

$$\frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu m r + \frac{\mu \zeta m r}{(1 - \beta \zeta)} = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\zeta \alpha}{m(1 - \beta \zeta + \zeta)} + 1 \right)_r}{\left(\frac{\zeta \alpha}{m(1 - \beta \zeta + \zeta)} \right)_r} \quad (9.1.26)$$

$$\frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu(\beta + 1)r + \frac{\mu \zeta m r}{(1 - \beta \zeta)} = \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \frac{\left(\frac{\zeta \alpha}{[(\beta + 1)(1 - \beta \zeta) + m \zeta]} + 1 \right)_r}{\left(\frac{\zeta \alpha}{[(\beta + 1)(1 - \beta \zeta) + m \zeta]} \right)_r} \quad (9.1.27)$$

$$\lambda + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu m r + \frac{\mu \zeta m (\beta + 1) r}{(1 - \beta \zeta)} = \frac{[\lambda(1 - \beta \zeta) + \mu \zeta \alpha]}{(1 - \beta \zeta)} \cdot \frac{\left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu m (\zeta + 1)} + 1 \right)_r}{\left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu m (\zeta + 1)} \right)_r} \quad (9.1.28)$$

$$\begin{aligned} \lambda + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu(\beta + 1)r + \frac{\mu \zeta m (\beta + 1) r}{(1 - \beta \zeta)} \\ = \frac{[\lambda(1 - \beta \zeta) + \mu \zeta \alpha]}{(1 - \beta \zeta)} \frac{\left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu(\beta + 1)(1 - \beta \zeta + m \zeta)} + 1 \right)_r}{\left(\frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\mu(\beta + 1)(1 - \beta \zeta + m \zeta)} \right)_r} \end{aligned} \quad (9.1.29)$$

9.2 Generating Relations for Generalized Polynomials $S_n^{(\alpha, \beta)}(x; m)$

Let the polynomial $S_n^{(\alpha, \beta)}(x; m)$ be defined by equation (9.1.1). Then

First General Result:-

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\ = (1 + \zeta)^\alpha \left\{ \lambda \sum_{n=0}^{\infty} \frac{1}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \frac{\mu}{(1 - \beta \zeta)} \right. \\ \left. \sum_{n=0}^{\infty} \frac{\{\zeta \alpha + m(1 + \zeta)n\}}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \right\} \end{aligned} \quad (9.2.1)$$

Second General Result:-

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\ = \frac{(1 + \zeta)^\alpha}{(1 - \beta \zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[(\lambda + \mu m n)(1 - \beta \zeta) + \mu \zeta(\beta + 1)m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \mu \zeta \alpha \right. \\ \left. \sum_{n=0}^{\infty} \frac{1}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \right\} \end{aligned} \quad (9.2.2)$$

Third General Result:-

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n$$

$$\begin{aligned}
&= \frac{(1+\zeta)^\alpha}{(1-\beta\zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[(\lambda + \mu m n)(1 - \beta\zeta) + \mu \zeta \beta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \mu \zeta \right. \\
&\quad \left. \sum_{n=0}^{\infty} \frac{(\alpha + m n)}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \right\} \quad (9.2.3)
\end{aligned}$$

Fourth General Result:-

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\
&= \frac{(1+\zeta)^\alpha}{(1-\beta\zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\lambda (1 - \beta\zeta) + \mu \zeta (\beta + 1) m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \right. \\
&\quad \left. \sum_{n=0}^{\infty} \frac{[\mu \zeta \alpha + \mu (1 - \beta\zeta) m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \right\} \quad (9.2.4)
\end{aligned}$$

Fifth General Result:-

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\
&= \frac{(1+\zeta)^\alpha}{(1-\beta\zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[(\lambda + \mu m n)(1 - \beta\zeta) + \mu \zeta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \right. \\
&\quad \left. \sum_{n=0}^{\infty} \frac{[\mu \zeta \alpha + \mu \beta \zeta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \right\} \quad (9.2.5)
\end{aligned}$$

Sixth General Result:-

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\
&= \frac{(1+\zeta)^\alpha}{(1-\beta\zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\lambda (1 - \beta\zeta) + \mu \zeta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \sum_{n=0}^{\infty} \frac{[\mu \zeta \alpha + \mu m n]}{\{\alpha + (\beta + 1)m n\}} \right. \\
&\quad \left. \gamma_n x^n \zeta^{m n} \right\} \quad (9.2.6)
\end{aligned}$$

Seventh General Result:- [117, p.391, Q.15 (iii); see also 109, p.331]

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\
&= (1+\zeta)^\alpha \left\{ \sum_{n=0}^{\infty} \frac{[\lambda + \mu m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \frac{\mu \zeta}{(1-\beta\zeta)} \sum_{n=0}^{\infty} \gamma_n x^n \zeta^{m n} \right\} \quad (9.2.7)
\end{aligned}$$

Eighth General Result:-

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\
&= \frac{(1 + \zeta)^\alpha}{(1 - \beta\zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\lambda(1 - \beta\zeta) + \mu\zeta\beta mn]}{\{\alpha + (\beta + 1)mn\}} \gamma_n x^n \zeta^{mn} + \right. \\
&\quad \left. \sum_{n=0}^{\infty} \frac{[\mu\zeta\alpha + \mu(1 - \beta\zeta + \zeta)mn]}{\{\alpha + (\beta + 1)mn\}} \gamma_n x^n \zeta^{mn} \right\} \quad (9.2.8)
\end{aligned}$$

Ninth General Result:-

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\
&= \frac{(1 + \zeta)^\alpha}{(1 - \beta\zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\lambda(1 - \beta\zeta) + \mu\zeta\alpha + \mu(\zeta + 1)mn]}{\{\alpha + (\beta + 1)mn\}} \gamma_n x^n \zeta^{mn} \right\} \quad (9.2.9)
\end{aligned}$$

where ζ is given by

$$\zeta = t(1 + \zeta)^{(\beta+1)}; \zeta(0) = 0 \quad (9.2.10)$$

provided that each of the series involved is absolutely convergent.

9.2.1 Independent Demonstrations of (9.2.1) to (9.2.9):

Independent Demonstration of (9.2.1):

Using the definition (9.1.1) of $S_n^{(\alpha, \beta)}(x; m)$ in left hand side of equation (9.2.1), we get

$$\begin{aligned}
\Omega^{**} &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(\lambda + \mu n) \Gamma\{\alpha + (\beta + 1)n + 1\}}{\{\alpha + (\beta + 1)n\} \Gamma(n - mr + 1) \Gamma\{\alpha + \beta n + mr + 1\}} \gamma_r x^r t^n \quad (9.2.11)
\end{aligned}$$

Applying summation identity (2.1.10) and then simplifying further, we get

$$\Omega^{**} = \sum_{r=0}^{\infty} \frac{\gamma_r x^r (t)^{mr}}{\{\alpha + m(\beta + 1)r\}} \left\{ \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\}(\lambda + \mu n + \mu mr)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \right\}$$

$$\left(\binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \right) \quad (9.2.12)$$

Now using first modified Gould's identity (2.1.7) with conditions (9.2.10), we get

$$\Omega^{**} = (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \left[\lambda + \left\{ \mu mr + \frac{\mu \zeta \{\alpha + (\beta + 1)mr\}}{(1 - \beta \zeta)} \right\} \right] \frac{\gamma_r x^r (\zeta)^{mr}}{\{\alpha + m(\beta + 1)r\}} \quad (9.2.13)$$

Changing the summation index from r to n and after solving it further, we get the general result (9.2.1) subject to the conditions (9.2.10).

Independent Demonstration of (9.2.7):

Using the definition (9.1.1) of $S_n^{(\alpha, \beta)}(x; m)$ in left hand side of equation (9.2.7), we get

$$\begin{aligned} \Omega^{**} &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(\lambda + \mu n) \Gamma\{\alpha + (\beta + 1)n + 1\}}{\{\alpha + (\beta + 1)n\} \Gamma(n - mr + 1) \Gamma\{\alpha + \beta n + mr + 1\}} \gamma_r x^r t^n \end{aligned} \quad (9.2.14)$$

Applying summation identity (2.1.10) and then simplifying further, we get

$$\Omega^{**} = \sum_{r=0}^{\infty} \frac{\gamma_r x^r (t)^{mr}}{\{\alpha + m(\beta + 1)r\}} \left\{ \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\}(\lambda + \mu n + \mu mr)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \right. \\ \left. \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \right\} \quad (9.2.15)$$

Now using first modified Gould's identity (2.1.7) with conditions (9.2.10), we get

$$\Omega^{**} = (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \left[\lambda + \mu mr + \left\{ \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \frac{\mu \zeta (\beta + 1)mr}{(1 - \beta \zeta)} \right\} \right] \frac{\gamma_r x^r (\zeta)^{mr}}{\{\alpha + m(\beta + 1)r\}} \quad (9.2.16)$$

Changing the summation index from r to n and after solving it further, we get the general result (9.2.7) subject to the conditions (9.2.10).

Independent Demonstration of (9.2.9):

Using the definition (9.1.1) of $S_n^{(\alpha, \beta)}(x; m)$ in left hand side of equation (9.2.9), we get

$$\begin{aligned} \Omega^{**} &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} S_n^{(\alpha, \beta)}(x; m) t^n \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{(\lambda + \mu n) \Gamma\{\alpha + (\beta + 1)n + 1\}}{\{\alpha + (\beta + 1)n\} \Gamma(n - mr + 1) \Gamma\{\alpha + \beta n + mr + 1\}} \gamma_r x^r t^n \end{aligned} \quad (9.2.17)$$

Applying summation identity (2.1.10) and then simplifying further, we get

$$\Omega^{**} = \sum_{r=0}^{\infty} \frac{\gamma_r x^r (t)^{mr}}{\{\alpha + m(\beta + 1)r\}} \left\{ \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\}(\lambda + \mu n + \mu mr)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \right. \\ \left. \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \right\} \quad (9.2.18)$$

Now using first modified Gould's identity (2.1.7) with conditions (9.2.10), we get

$$\Omega^{**} = (1+\zeta)^\alpha \sum_{r=0}^{\infty} \left[\left\{ \lambda + \frac{\mu\zeta\alpha}{(1-\beta\zeta)} \right\} + \left\{ \mu mr + \frac{\mu\zeta(\beta + 1)mr}{(1-\beta\zeta)} \right\} \right] \frac{\gamma_r x^r (\zeta)^{mr}}{\{\alpha + m(\beta + 1)r\}} \quad (9.2.19)$$

Changing the summation index from r to n and after solving it further, we get the general result (9.2.9) subject to the conditions (9.2.10).

Proceeding in the similar way, we can derive the remaining six results from (9.2.2) to (9.2.6) and (9.2.8).

9.3 Generating Relations For Another Generalized Polynomials $T_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$

Let the polynomial $T_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ be given by (9.1.2). Then

First General Result:-

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ = (1 + \zeta)^\alpha \left\{ \lambda \sum_{n=0}^{\infty} \frac{1}{\{\alpha + (\beta + 1)mn\}} \gamma_n x^n \zeta^{mn} + \frac{\mu}{(1 - \beta\zeta)} \right. \\ \left. \sum_{n=0}^{\infty} \frac{[\zeta\alpha + (\beta + 1)(1 - \beta\zeta + m\zeta)n]}{\{\alpha + (\beta + 1)mn\}} \gamma_n x^n \zeta^{mn} \right\} \quad (9.3.1)$$

Second General Result:-

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ = \frac{(1 + \zeta)^\alpha}{(1 - \beta\zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\{\lambda + \mu(\beta + 1)n\}(1 - \beta\zeta) + \mu\zeta(\beta + 1)mn]}{\{\alpha + (\beta + 1)mn\}} \gamma_n x^n \zeta^{mn} \right.$$

$$+\mu \zeta^\alpha \sum_{n=0}^{\infty} \frac{1}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \Big\} \quad (9.3.2)$$

Third General Result:-

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= \frac{(1 + \zeta)^\alpha}{(1 - \beta \zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\{\lambda + \mu (\beta + 1)n\}(1 - \beta \zeta) + \mu \zeta \beta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \mu \zeta \right. \\ & \quad \left. \sum_{n=0}^{\infty} \frac{(\alpha + m n)}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \right\} \end{aligned} \quad (9.3.3)$$

Fourth General Result:-

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= \frac{(1 + \zeta)^\alpha}{(1 - \beta \zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\lambda (1 - \beta \zeta) + \mu \zeta (\beta + 1)m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \right. \\ & \quad \left. \sum_{n=0}^{\infty} \frac{[\mu \zeta \alpha + \mu (1 - \beta \zeta) (\beta + 1)n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \right\} \end{aligned} \quad (9.3.4)$$

Fifth General Result:-

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= \frac{(1 + \zeta)^\alpha}{(1 - \beta \zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\{\lambda + \mu (\beta + 1)n\}(1 - \beta \zeta) + \mu \zeta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \right. \\ & \quad \left. \sum_{n=0}^{\infty} \frac{[\mu \zeta (\alpha + \beta m n)]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \right\} \end{aligned} \quad (9.3.5)$$

Sixth General Result:-

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= \frac{(1 + \zeta)^\alpha}{(1 - \beta \zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\lambda (1 - \beta \zeta) + \mu \zeta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \right. \end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{[\mu \zeta \alpha + \mu (\beta + 1)(1 - \beta \zeta) n + \mu \zeta \beta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \left\} \quad (9.3.6)$$

Seventh General Result:-

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= (1 + \zeta)^\alpha \left\{ \sum_{n=0}^{\infty} \frac{[\lambda + \mu (\beta + 1) n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \frac{\mu \zeta}{(1 - \beta \zeta)} \sum_{n=0}^{\infty} \gamma_n x^n \zeta^{m n} \right\} \quad (9.3.7) \end{aligned}$$

Eighth General Result:-

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= \frac{(1 + \zeta)^\alpha}{(1 - \beta \zeta)} \left\{ \sum_{n=0}^{\infty} \frac{[\lambda (1 - \beta \zeta) + \mu \zeta \beta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} + \right. \\ & \quad \left. \sum_{n=0}^{\infty} \frac{[\mu \zeta \alpha + \mu (1 - \beta \zeta)(\beta + 1)n + \mu \zeta m n]}{\{\alpha + (\beta + 1)m n\}} \gamma_n x^n \zeta^{m n} \right\} \quad (9.3.8) \end{aligned}$$

Ninth General Result:-

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= \frac{(1 + \zeta)^\alpha}{(1 - \beta \zeta)} \sum_{n=0}^{\infty} \left[\frac{\{\lambda (1 - \beta \zeta) + \mu \zeta \alpha + \mu (1 + \beta)(1 - \beta \zeta + m \zeta) n\}}{\{\alpha + (\beta + 1)m n\}} \right] \gamma_n x^n \zeta^{m n} \quad (9.3.9) \end{aligned}$$

where ζ is given by (9.2.10):

$$\zeta = t(1 + \zeta)^{(\beta+1)}; \zeta(0) = 0$$

provided that each of the series involved is absolutely convergent.

9.3.1 Independent Demonstrations of (9.3.1) to (9.3.9):

Independent Demonstrations of (9.3.1):

Using the definition (9.1.2) of $T_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ in left hand side of equation (9.3.1), we get

$$\Omega^{***} = \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \left\{ \binom{\alpha + (\beta + 1)n}{n - mr} \frac{\{\lambda + \mu n + \mu(\beta + 1)r - \mu mr\}}{\{\alpha + (\beta + 1)n\}} \right\} \gamma_r x^r t^n \quad (9.3.10)$$

Applying summation identity (2.1.10) and then simplifying further, we get

$$\Omega^{***} = \sum_{r=0}^{\infty} \frac{\gamma_r x^r (t)^{mr}}{\{\alpha + m(\beta + 1)r\}} \left\{ \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} \{\lambda + \mu n + \mu(\beta + 1)r\}}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \right. \\ \left. \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \right\} \quad (9.3.11)$$

Now using second modified Gould's identity (2.1.8) with conditions (9.2.10), we get

$$\Omega^{***} = (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \left[\lambda + \left\{ \mu(\beta + 1)r + \frac{\mu\zeta \{\alpha + (\beta + 1)mr\}}{(1 - \beta\zeta)} \right\} \right] \cdot \frac{\gamma_r x^r (\zeta)^{mr}}{\{\alpha + m(\beta + 1)r\}} \quad (9.3.12)$$

Changing the summation index from r to n and after solving it further, we get the general result (9.3.1) subject to the conditions (9.2.10).

Independent Demonstration of (9.3.7):

Using the definition (9.1.2) of $T_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ in left hand side of equation (9.3.7), we get

$$\Omega^{***} = \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ = \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \left\{ \binom{\alpha + (\beta + 1)n}{n - mr} \frac{\{\lambda + \mu n + \mu(\beta + 1)r - \mu mr\}}{\{\alpha + (\beta + 1)n\}} \right\} \gamma_r x^r t^n \quad (9.3.13)$$

Applying summation identity (2.1.10) and then simplifying further, we get

$$\Omega^{***} = \sum_{r=0}^{\infty} \frac{\gamma_r x^r (t)^{mr}}{\{\alpha + m(\beta + 1)r\}} \left\{ \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} \{\lambda + \mu n + \mu(\beta + 1)r\}}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \right. \\ \left. \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \right\} \quad (9.3.14)$$

Now using second modified Gould's identity (2.1.8) with conditions (9.2.10), we get

$$\Omega^{***} = (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \left[\{\lambda + \mu(\beta + 1)r\} + \left\{ \frac{\mu\zeta\alpha}{(1 - \beta\zeta)} + \frac{\mu\zeta(\beta + 1)mr}{(1 - \beta\zeta)} \right\} \right] \cdot \frac{\gamma_r x^r (\zeta)^{mr}}{\{\alpha + m(\beta + 1)r\}} \quad (9.3.15)$$

Changing the summation index from r to n and after solving it further, we get the general result (9.3.7) subject to the conditions (9.2.10).

Independent Demonstration of (9.3.9):

Using the definition (9.1.2) of $T_n^{(\alpha,\beta)}(x; m, \lambda, \mu)$ in left hand side of equation (9.3.9), we get

$$\begin{aligned}\Omega^{***} &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} T_n^{(\alpha,\beta)}(x; m, \lambda, \mu) t^n \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \left[\binom{\alpha + (\beta + 1)n}{n - mr} \frac{\{\lambda + \mu n + \mu(\beta + 1)r - \mu mr\}}{\{\alpha + (\beta + 1)n\}} \right] \gamma_r x^r t^n \quad (9.3.16)\end{aligned}$$

Applying summation identity (2.1.10) and then simplifying further, we get

$$\begin{aligned}\Omega^{***} &= \sum_{r=0}^{\infty} \frac{\gamma_r x^r (t)^{mr}}{\{\alpha + m(\beta + 1)r\}} \left\{ \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} \{\lambda + \mu n + \mu(\beta + 1)r\}}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \right. \\ &\quad \left. \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \right\} \quad (9.3.17)\end{aligned}$$

Now using second modified Gould's identity (2.1.8) with conditions (9.2.10), we get

$$\begin{aligned}\Omega^{***} &= (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \left[\left\{ \lambda + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} \right\} + \left\{ \mu(\beta + 1)r + \frac{\mu \zeta (\beta + 1)mr}{(1 - \beta \zeta)} \right\} \right] \\ &\quad \frac{\gamma_r x^r (\zeta)^{mr}}{\{\alpha + m(\beta + 1)r\}} \quad (9.3.18)\end{aligned}$$

Changing the summation index from r to n and after solving it further, we get the general result (9.3.9) subject to the conditions (9.2.10).

Proceeding in the similar way, we can derive the remaining six results from (9.3.2) to (9.3.6) and (9.3.8).

In the definitions of generalized polynomials given by $S_n^{(\alpha,\beta)}(x; m)$ and $T_n^{(\alpha,\beta)}(x; m, \lambda, \mu)$, putting

$$\gamma_r = \frac{(-1)^{mr} (a_1)_r \dots (a_p)_r}{(b_1)_r \dots (b_q)_r r!},$$

we obtain Srivastava's generalized hypergeometric polynomials of one variable $H_n^{(\alpha,\beta)}(x; m)$ and Pathan's generalized hypergeometric polynomials of one variable $\mathcal{B}_n^{(\alpha,\beta)}(x; m, \lambda, \mu)$ respectively.

9.4 Generating Relations Involving Hypergeometric Polynomials $H_n^{(\alpha, \beta)}(x; m)$

Setting $\gamma_n = \frac{(-1)^m n (a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n n!}$, in section 9.2, we get the following generating relations. If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

First Generating Relation: [see eq. (4.2.1)]

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p) \\ 1 + \frac{\alpha}{m(\beta+1)}, (b_q) \end{matrix}; x(-\zeta)^m \right] \right. \\ & \quad \left. + \frac{\mu\zeta}{(1 - \beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p), 1 + \frac{\alpha\zeta}{m(\zeta+1)} \\ 1 + \frac{\alpha}{m(\beta+1)}, (b_q), \frac{\alpha\zeta}{m(\zeta+1)} \end{matrix}; x(-\zeta)^m \right] \right\} \end{aligned} \quad (9.4.1)$$

Second Generating Relation:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{m\mu(1+\zeta)}, (a_p) \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{m\mu(1+\zeta)}, (b_q) \end{matrix}; x(-\zeta)^m \right] \right. \\ & \quad \left. + \frac{\mu\zeta}{(1 - \beta\zeta)} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p) \\ \frac{\alpha}{m(\beta+1)} + 1, (b_q) \end{matrix}; x(-\zeta)^m \right] \right\} \end{aligned} \quad (9.4.2)$$

Third Generating Relation:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{m\mu}, (a_p) \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{m\mu}, (b_q) \end{matrix}; x(-\zeta)^m \right] \right. \\ & \quad \left. + \frac{\mu\zeta}{(1 - \beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\alpha}{m} + 1, (a_p) \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\alpha}{m}, (b_q) \end{matrix}; x(-\zeta)^m \right] \right\} \end{aligned} \quad (9.4.3)$$

Fourth Generating Relation:

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n \\
&= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{m\mu\zeta(\beta+1)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{m\mu\zeta(\beta+1)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\
& \quad \left. + \frac{\mu\zeta}{(1-\beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\alpha\zeta}{m(1-\beta\zeta)} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\alpha\zeta}{m(1-\beta\zeta)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.4.4)
\end{aligned}$$

Fifth Generating Relation:

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n \\
&= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{m\mu(1-\beta\zeta+\zeta)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{m\mu(1-\beta\zeta+\zeta)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\
& \quad \left. + \frac{\mu\zeta}{(1-\beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\alpha}{m\beta} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\alpha}{m\beta}, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.4.5)
\end{aligned}$$

Sixth Generating Relation:

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n \\
&= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{m\mu\zeta}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{m\mu\zeta}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\
& \quad \left. + \frac{\mu\zeta}{(1-\beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\alpha\zeta}{m} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\alpha\zeta}{m}, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.4.6)
\end{aligned}$$

Seventh Generating Relation: [see 117, Q.16 and 109, p. 332][eq.(3.2.1)]

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n = (1 + \zeta)^\alpha. \\
& \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda}{m\mu}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda}{m\mu}, (b_q); \end{matrix} x(-\zeta)^m \right] + \frac{\mu\zeta}{(1-\beta\zeta)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.4.7)
\end{aligned}$$

Eighth Generating Relation:

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n \\
&= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{m\mu\zeta\beta}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{m\mu\zeta\beta}, (b_q); \end{matrix} \right. x(-\zeta)^m \right] \right. \\
& \quad \left. + \frac{\mu\zeta}{(1-\beta\zeta)^{p+2}} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\alpha\zeta}{m(1-\beta\zeta+\zeta)} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\alpha\zeta}{m(1-\beta\zeta+\zeta)}, (b_q); \end{matrix} \right. x(-\zeta)^m \right] \right\} \quad (9.4.8)
\end{aligned}$$

Ninth Generating Relation:[see eq.(2.2.1)]

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n = (1 + \zeta)^\alpha \cdot \left\{ \frac{\lambda(1 - \beta\zeta) + \mu\zeta\alpha}{\alpha(1 - \beta\zeta)} \right\} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta) + \mu\zeta\alpha}{m\mu(1+\zeta)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta) + \mu\zeta\alpha}{m\mu(1+\zeta)}, (b_q); \end{matrix} \right. x(-\zeta)^m \right] \\
& \hspace{15cm} (9.4.9)
\end{aligned}$$

where

$$\zeta = t(1 + \zeta)^{1+\beta}; \zeta(0) = 0.$$

provided that involved series on both sides are absolutely convergent.

Here Srivastava's generalized hypergeometric polynomials $H_n^{(\alpha, \beta)}(x; m)$ [17, p.360, eq.(7.3.3)] are given by equation (2.2.2).

Independent Demonstration:

Using the definition (2.2.2) of $H_n^{(\alpha, \beta)}(x; m)$ and then the power series form of ${}_{p+m}F_{q+m}[x]$ in left hand side of equation (9.4.2), we get

$$\begin{aligned}
\Omega &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} H_n^{(\alpha, \beta)}(x; m) t^n = \sum_{n=0}^{\infty} \frac{(\lambda + \mu n) \Gamma\{\alpha + (\beta + 1)n + 1\}}{\{\alpha + (\beta + 1)n\} \Gamma(n + 1) \Gamma\{\alpha + \beta n + 1\}} \cdot \\
& \quad \cdot \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{\prod_{j=1}^m \left(\frac{-n + j - 1}{m} \right)_r [(a_p)]_r x^r t^n}{\prod_{j=1}^m \left(\frac{1 + \alpha + n\beta + j - 1}{m} \right)_r [(b_q)]_r r!}
\end{aligned}$$

Using Gauss's multiplication theorem (2.1.9) in above equation, we get

$$\Omega = \sum_{n=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{(\lambda + \mu n) (-n)_{mr} (\alpha)_{n(\beta+1)} [(a_p)]_r x^r t^n}{\alpha (\alpha + 1)_{n\beta} (1 + \alpha + n\beta)_{mr} [(b_q)]_r r! n!} \quad (9.4.10)$$

Now applying summation identity (2.1.10) and then simplifying further, we get

$$\Omega = \sum_{r=0}^{\infty} \frac{(\alpha)_{m(\beta+1)r} [(a_p)]_r x^r (-t)^{mr}}{\alpha (\alpha+1)_{m(\beta+1)r} [(b_q)]_r r!} \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta+1)mr\}(\lambda + \mu n + \mu mr)}{\{\alpha + (\beta+1)mr + (\beta+1)n\}} \cdot \binom{\alpha + (\beta+1)mr + (\beta+1)n}{n} t^n \quad (9.4.11)$$

Now using first modified Gould's identity (2.1.7), we get

$$\Omega = (1+\zeta)^\alpha \sum_{r=0}^{\infty} \frac{(\alpha)_{m(\beta+1)r} [(a_p)]_r x^r (-t)^{mr}}{\alpha (\alpha+1)_{m(\beta+1)r} [(b_q)]_r r!} (1+\zeta)^{m(\beta+1)r} \cdot \left[\frac{\mu\zeta\alpha}{(1-\beta\zeta)} + \left(\lambda + \mu mr + \frac{\mu\zeta(\beta+1)mr}{(1-\beta\zeta)} \right) \right] \quad (9.4.12)$$

Simplifying it further, we get

$$\Omega = (1+\zeta)^\alpha \sum_{r=0}^{\infty} \frac{\left(\frac{\alpha}{m(\beta+1)}\right)_r [(a_p)]_r x^r (-\zeta)^{mr}}{\alpha \left(\frac{\alpha}{m(\beta+1)} + 1\right)_r [(b_q)]_r r!} \cdot \left[\lambda \frac{\left(\frac{\lambda(1-\beta\zeta)}{\mu m(1+\zeta)} + 1\right)_r}{\left(\frac{\lambda(1-\beta\zeta)}{\mu m(1+\zeta)}\right)_r} + \frac{\mu\zeta\alpha}{(1-\beta\zeta)} \right] \quad (9.4.13)$$

After solving it further, we get the result (9.4.2) in the form of sum of two generalized hypergeometric functions of one variable. We can obtain the remaining five generating relations from (9.4.3) to (9.4.6) and (9.4.8) following the same procedure and taking different combinations of terms of $\left[\lambda + \mu mr + \frac{\mu\zeta\alpha}{(1-\beta\zeta)} + \frac{\mu\zeta\beta mr}{(1-\beta\zeta)} + \frac{\mu\zeta mr}{(1-\beta\zeta)} \right]$. Derivations of (9.4.1), (9.4.7) and (9.4.9) are already mentioned in chapters 4, 3, 2 respectively.

9.5 Generating Relations Involving Hypergeometric Polynomials $\mathcal{B}_n^{(\alpha,\beta)}(x; m, \lambda, \mu)$

Setting $\gamma_n = \frac{(-1)^m n (a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n n!}$, in section 9.3, we get the following generating relations. If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

First Generating Relation:[see eq.(4.2.7)]

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta+1)n\}} \mathcal{B}_n^{(\alpha,\beta)}(x; m, \lambda, \mu) t^n = (1+\zeta)^\alpha \left\{ \frac{\lambda}{\alpha^{p+1} F_{q+1}} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p) & ; \\ 1 + \frac{\alpha}{m(\beta+1)}, (b_q) & ; \end{matrix} x(-\zeta)^m \right] \right.$$

$$+ \frac{\mu\zeta}{(1-\beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\zeta\alpha}{(\beta+1)(1+m\zeta-\beta\zeta)} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\zeta\alpha}{(\beta+1)(1+m\zeta-\beta\zeta)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.5.1)$$

Second Generating Relation:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{\mu(\beta+1)(1-\beta\zeta+m\zeta)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{\mu(\beta+1)(1-\beta\zeta+m\zeta)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\ & \quad \left. + \frac{\mu\zeta}{(1-\beta\zeta)} {}_{p+1}F_{q+1} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, (a_p) & ; \\ \frac{\alpha}{m(\beta+1)} + 1, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.5.2) \end{aligned}$$

Third Generating Relation:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{\mu[\beta+1+\beta\zeta(m-\beta-1)]}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{\mu[\beta+1+\beta\zeta(m-\beta-1)]}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\ & \quad \left. + \frac{\mu\zeta}{(1-\beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\alpha}{m} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\alpha}{m}, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.5.3) \end{aligned}$$

Fourth Generating Relation:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{\mu m \zeta(\beta+1)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{\mu m \zeta(\beta+1)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\ & \quad \left. + \frac{\mu\zeta}{(1-\beta\zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\alpha\zeta}{(1-\beta\zeta)(\beta+1)} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\alpha\zeta}{(1-\beta\zeta)(\beta+1)}, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.5.4) \end{aligned}$$

Fifth Generating Relation:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{\mu\{(\beta+1)(1-\beta\zeta)+m\zeta\}}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{\mu\{(\beta+1)(1-\beta\zeta)+m\zeta\}}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \end{aligned}$$

$$+ \frac{\mu \zeta}{(1 - \beta \zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\alpha}{m\beta} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\alpha}{m\beta}, (b_q); \end{matrix} x(-\zeta)^m \right] \Bigg\} \quad (9.5.5)$$

Sixth Generating Relation:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{\mu m \zeta}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{\mu m \zeta}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\ & \left. + \frac{\mu \zeta}{(1 - \beta \zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\zeta \alpha}{\{(\beta+1)(1-\beta\zeta)+\beta m \zeta\}} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\zeta \alpha}{\{(\beta+1)(1-\beta\zeta)+\beta m \zeta\}}, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.5.6) \end{aligned}$$

Seventh Generating Relation: [see 117, p.391, Q.16; see also 109, p.332][eq.(3.2.7)]

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n = (1 + \zeta)^\alpha \cdot \\ & \cdot \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda}{\mu(\beta+1)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda}{\mu(\beta+1)}, (b_q); \end{matrix} x(-\zeta)^m \right] + \frac{\mu \zeta}{(1 - \beta \zeta)} {}_pF_q \left[\begin{matrix} (a_p); \\ (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.5.7) \end{aligned}$$

Eighth Generating Relation:

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n \\ &= (1 + \zeta)^\alpha \left\{ \frac{\lambda}{\alpha} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)}{\mu \beta m \zeta}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)}{\mu \beta m \zeta}, (b_q); \end{matrix} x(-\zeta)^m \right] \right. \\ & \left. + \frac{\mu \zeta}{(1 - \beta \zeta)} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, \frac{\zeta \alpha}{\{(\beta+1)(1-\beta\zeta)+m \zeta\}} + 1, (a_p); \\ \frac{\alpha}{m(\beta+1)} + 1, \frac{\zeta \alpha}{\{(\beta+1)(1-\beta\zeta)+m \zeta\}}, (b_q); \end{matrix} x(-\zeta)^m \right] \right\} \quad (9.5.8) \end{aligned}$$

Ninth Generating Relation:[see eq.(2.2.7)]

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n = (1 + \zeta)^\alpha \cdot \\ & \cdot \left\{ \frac{\lambda(1 - \beta \zeta) + \mu \zeta \alpha}{\alpha(1 - \beta \zeta)} \right\} {}_{p+2}F_{q+2} \left[\begin{matrix} \frac{\alpha}{m(\beta+1)}, 1 + \frac{\lambda(1-\beta\zeta)+\mu\zeta\alpha}{\mu(1+\beta)(1-\beta\zeta+m\zeta)}, (a_p); \\ 1 + \frac{\alpha}{m(\beta+1)}, \frac{\lambda(1-\beta\zeta)+\mu\zeta\alpha}{\mu(1+\beta)(1-\beta\zeta+m\zeta)}, (b_q); \end{matrix} x(-\zeta)^m \right] \quad (9.5.9) \end{aligned}$$

where

$$\zeta = t(1 + \zeta)^{1+\beta}; \zeta(0) = 0.$$

provided that involved series on both sides are absolutely convergent.

Here Pathan's generalized Hypergeometric polynomials $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ are given by equation (2.2.8).

Independent Demonstration:

Using the definition (2.2.8) of $\mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu)$ and then the power series form of ${}_{p+m+1}F_{q+m+1}[x]$ in left hand side of equation (9.5.2), using Gauss's multiplication theorem (2.1.9) and result (9.1.5), we get

$$\begin{aligned} \Omega^* &= \sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta + 1)n\}} \mathcal{B}_n^{(\alpha, \beta)}(x; m, \lambda, \mu) t^n = \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{[\lambda + \mu n + \mu(\beta + 1)r - \mu mr] (-n)_{mr} \Gamma\{\alpha + n(\beta + 1)\} [(a_p)]_r x^r t^n}{\Gamma\{\alpha + n\beta + 1 + mr\} [(b_q)]_r r! n!} \end{aligned} \quad (9.5.10)$$

Now applying summation identity (2.1.10) in above equation then simplifying further, we get

$$\begin{aligned} \Omega^* &= \sum_{r=0}^{\infty} \frac{[(a_p)]_r x^r (-t)^{mr}}{[\alpha + m(\beta + 1)r] [(b_q)]_r r!} \sum_{n=0}^{\infty} \frac{\{\alpha + (\beta + 1)mr\} (\lambda + \mu n + \mu(\beta + 1)r)}{\{\alpha + (\beta + 1)mr + (\beta + 1)n\}} \\ &\quad \cdot \binom{\alpha + (\beta + 1)mr + (\beta + 1)n}{n} t^n \end{aligned} \quad (9.5.11)$$

Now using second modified Gould's identity (2.1.8), we get

$$\begin{aligned} \Omega^* &= (1 + \zeta)^\alpha \sum_{r=0}^{\infty} \frac{[(a_p)]_r x^r (-t)^{mr} (1 + \zeta)^{m(\beta + 1)r}}{[(b_q)]_r \{\alpha + m(\beta + 1)r\} r!} \\ &\quad \cdot \left[\frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \left\{ \lambda + \mu(\beta + 1)r + \frac{\mu \zeta (\beta + 1)mr}{(1 - \beta \zeta)} \right\} \right] \end{aligned} \quad (9.5.12)$$

Now using equations (9.1.6) and (9.1.10) in above equation, then summing it up into hypergeometric form further, we get the desired result (9.5.2).

We can obtain the remaining five generating relations from (9.5.3) to (9.5.6) and (9.5.8) following the same procedure and taking different combinations of terms of $\left[\lambda + \frac{\mu \zeta \alpha}{(1 - \beta \zeta)} + \mu(\beta + 1)r + \frac{\mu \zeta \beta mr}{(1 - \beta \zeta)} + \frac{\mu \zeta mr}{(1 - \beta \zeta)} \right]$. Derivations of (9.5.1), (9.5.7) and (9.5.9) are already mentioned in chapters 4, 3, 2 respectively.

I hope that the investigations carried over in the present thesis on multiple hypergeometric series are of some interest and will provide insight into the structure of special functions in general and hypergeometric series and functions in particular and would motivate further research. It has been our aim to derive new results and to show that there is a fruitful interaction and connection between the hypergeometric functions of different types and phenomena nature. At the same time, we demonstrate that multiple hypergeometric series possesses a special character and has particular applications, which clearly makes it not merely a subtopic of curious generalization (in the study of hypergeometric series), but a field in itself.

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Appendix 1.

Reprints of Research Papers

Published in the Journals

Appendix 2.

Certificates of Conferences

Attended and Papers Presented