

Hypergeometric Transformations Involving Terminating Clausen's Series Using Beta Integral Technique

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Abstract: The main object of this paper is to obtain the correct form of two erroneous integrals of Erdélyi et al. [*Tables of Integral Transforms*, Vol.II, p.284(16.4.1 and 16.4.3)]. Five hypergeometric transformations of the terminating Clausen's series ${}_3F_2$ are also obtained.

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1 Introduction and Preliminaries

Throughout in the present paper, we use the following standard notations:

$\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \dots\} = \mathbb{Z}_0^- \setminus \{0\}$.

Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \dots (\lambda + \nu - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases} \quad (1.1)$$

it being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

In the Gaussian hypergeometric series ${}_2F_1(a, b; c; z)$ there are two numerator parameters a, b and one denominator parameter c . A natural generalization of this series is accomplished by introducing any arbitrary number of numerator and denominator parameters. The resulting series

$${}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} \frac{z^n}{n!} = {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \quad (1.2)$$

where $(\lambda)_n$ is the pochhammer symbol defined by (1.1), is known as the *generalized Gauss series*, or simply, the *generalized hypergeometric series*. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z , the numerator parameters $\alpha_1, \dots, \alpha_p$ and the denominator parameters $\beta_1, \dots, \beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \dots; \quad j = 1, \dots, q. \quad (1.3)$$

Supposing that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the usual restriction (1.3), the ${}_pF_q$ series in (1.2)

- (i) converges for $|z| < \infty$ if $p \leq q$
- (ii) converges for $|z| < 1$ if $p = q + 1$ and
- (i) diverges for all z , $z \neq 0$, if $p > q + 1$.

Furthermore, if we set

$$\omega = \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j. \quad (1.4)$$

It is known that the ${}_pF_q$ series, with $p = q + 1$, is

- I. absolutely convergent for $|z| = 1$ if $\operatorname{Re}(\omega) > 0$,
- II. conditionally convergent for $|z| = 1, z \neq 1$, if $-1 < \operatorname{Re}(\omega) \leq 0$, and
- III. divergent for $|z| = 1$ if $\operatorname{Re}(\omega) \leq -1$.

The Jacobi Polynomials of first kind $P_n^{(\alpha, \beta)}(x)$ are defined by [2, p.254 (132.1, 132.2, 132.3), p.255 (132.7, 132.8, 132.9)].

$$P_n^{(\alpha, \beta)}(x) = \frac{(1 + \alpha)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1 + \alpha + \beta + n; \\ 1 + \alpha \end{matrix} ; \frac{1 - x}{2} \right] \quad (1.5)$$

$$= \frac{(1+\alpha)_n}{n!} \left(\frac{x+1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -\beta-n & ; & \frac{x-1}{x+1} \\ 1+\alpha & ; & \end{matrix} \right] \quad (1.6)$$

$$= \frac{(-1)^n (1+\beta)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, 1+\alpha+\beta+n & ; & \frac{1+x}{2} \\ 1+\beta & ; & \end{matrix} \right] \quad (1.7)$$

$$= \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n} \left(\frac{x-1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -\alpha-n & ; & \frac{2}{1-x} \\ -\alpha-\beta-2n & ; & \end{matrix} \right] \quad (1.8)$$

$$= \frac{(1+\beta)_n}{n!} \left(\frac{x-1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -\alpha-n & ; & \frac{x+1}{x-1} \\ 1+\beta & ; & \end{matrix} \right] \quad (1.9)$$

$$= \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n} \left(\frac{x+1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -\beta-n & ; & \frac{2}{x+1} \\ -\alpha-\beta-2n & ; & \end{matrix} \right], \quad (1.10)$$

where n is a non-negative integer.

An integral reducible to Beta function [2, p.31Q.No.7]

$$\int_{-1}^{+1} (1-x)^{p-1} (1+x)^{q-1} dx = 2^{p+q-1} B(p, q) \quad (1.11)$$

where $Re(p) > 0$, $Re(q) > 0$.

Gauss summation theorem [2, p.49 Th.18]

$${}_2F_1(A, B; C; 1) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} \quad (1.12)$$

where $Re(C-A-B) > 0$, and $C \neq 0, -1, -2, -3, \dots$

Saalschütz' summation theorem for terminating series [2, p.87 Th.29]

$${}_3F_2(-N, A, B; C, D; 1) = \frac{(C-A)_N (C-B)_N}{(C)_N (C-A-B)_N} \quad (1.13)$$

where $(C+D) - (-N+A+B) = 1$, and N is a non-negative integer.

Two definite integrals of Erdélyi et al.[1,p.284(16.4.1, 16.4.3)]

$$\int_{-1}^{+1} (1-x)^{\alpha}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\sigma+1}\Gamma(\sigma+1)\Gamma(\alpha+n+1)\Gamma(\sigma-\beta+1)}{\Gamma(\sigma-\beta-n+1)\Gamma(\alpha+\sigma+n+2)} \quad (1.14)$$

where $Re(\alpha) > -1$, $Re(\sigma) > -1$.

$$\begin{aligned} \int_{-1}^{+1} (1-x)^{\rho}(1+x)^{\sigma} P_n^{(\alpha,\beta)}(x)dx &= \frac{2^{\rho+\sigma+1}\Gamma(\rho+1)\Gamma(\sigma+1)}{\Gamma(\rho+\sigma+2)} \times \\ &\times {}_3F_2(-n, \alpha+\beta+n+1, \rho+1; \alpha+1, \rho+\sigma+2; 1) \end{aligned} \quad (1.15)$$

where $Re(\rho) > -1$, $Re(\sigma) > -1$.

Combinatorial identity

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = \sum_{k=0}^n \frac{(-n)_k}{k!} = \begin{cases} 0, & \text{if } n = 1, 2, 3, \dots \\ 1, & \text{if } n=0 \end{cases} \quad (1.16)$$

which can be verified easily and $\binom{n}{k}$ is binomial coefficient.

An important property of Jacobi polynomials [2,p.258(133.2)]

$$P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x) \quad (1.17)$$

Definite integral property

$$\int_{-1}^{+1} F(x)dx = \int_{-1}^{+1} F(-x)dx \quad (1.18)$$

2 Evaluation of the integral

$$\int_{-1}^{+1} (1-x)^a(1+x)^b P_n^{(\alpha,\beta)}(x)dx$$

If any values of parameters leading to the results which do not make sense, are tacitly excluded, then

$$\int_{-1}^{+1} (1-x)^a(1+x)^b P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{2^{a+b+1}\Gamma(b+1)\Gamma(a+1)}{n!} \frac{(1+\alpha)_n}{\Gamma(a+b+2)} {}_3F_2 \left[\begin{matrix} -n, \alpha + \beta + n + 1, a + 1; \\ 1 + \alpha, \quad a + b + 2; \end{matrix} \quad 1 \right] \quad (2.1)$$

$$= \frac{2^{a+b+1}\Gamma(a+1)\Gamma(b+n+1)(1+\alpha)_n}{n!\Gamma(a+b+n+2)} {}_3F_2 \left[\begin{matrix} -n, -\beta - n, a + 1; \\ 1 + \alpha, \quad -b - n; \end{matrix} \quad 1 \right] \quad (2.2)$$

$$= \frac{2^{a+b+1}\Gamma(a+1)\Gamma(b+1)(-1)^n(1+\beta)_n}{n!\Gamma(a+b+2)} {}_3F_2 \left[\begin{matrix} -n, 1 + \alpha + \beta + n, b + 1; \\ 1 + \beta, \quad a + b + 2; \end{matrix} \quad 1 \right] \quad (2.3)$$

$$= \frac{2^{a+b+1}(-1)^n\Gamma(b+1)\Gamma(a+n+1)(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n\Gamma(a+b+n+2)} {}_3F_2 \left[\begin{matrix} -n, -\alpha - n, -a - b - n - 1; \\ -\alpha - \beta - 2n, \quad -a - n; \end{matrix} \quad 1 \right] \quad (2.4)$$

$$= \frac{(-1)^n 2^{a+b+1}(1+\beta)_n\Gamma(b+1)\Gamma(a+n+1)}{n! \Gamma(a+b+n+2)} {}_3F_2 \left[\begin{matrix} -n, -\alpha - n, b + 1; \\ 1 + \beta, \quad -a - n; \end{matrix} \quad 1 \right] \quad (2.5)$$

$$= \frac{2^{a+b+1}\Gamma(a+1)\Gamma(b+n+1)(1+\alpha+\beta)_{2n}}{n! (1+\alpha+\beta)_n\Gamma(a+b+n+2)} {}_3F_2 \left[\begin{matrix} -n, -\beta - n, -a - b - n - 1; \\ -\alpha - \beta - 2n, \quad -b - n; \end{matrix} \quad 1 \right] \quad (2.6)$$

where $Re(a) > -1$, $Re(b) > -1$.

Proof Applying the definition (1.5) of Jacobi polynomials in the integral $\int_{-1}^{+1} (1-x)^a(1+x)^b P_n^{(\alpha, \beta)}(x) dx$, expanding hypergeometric polynomial ${}_2F_1$ in series form, interchanging the order of integration and summation, using the Beta integral (1.11) and then writing the resulting series in hypergeometric notation, we get result(2.1). Similarly using the remaining five definitions(1.6), (1.7), (1.8), (1.9) and (1.10) of Jacobi polynomials, we can evaluate the same integral in the forms of (2.2), (2.3), (2.4), (2.5) and (2.6) respectively.

3 Main Hypergeometric Transformations

If any values of parameters leading to the results which do not make sense, are tacitly excluded, then

$$\begin{aligned}
 & {}_3F_2 \left[\begin{matrix} -n, 1 + \alpha + \beta + n, 1 + a; \\ 1 + \alpha, a + b + 2 \end{matrix} ; 1 \right] \\
 &= \frac{(1+b)_n}{(a+b+2)_n} {}_3F_2 \left[\begin{matrix} -n, -\beta - n, 1 + a \\ 1 + \alpha, -b - n \end{matrix} ; 1 \right] \tag{3.1}
 \end{aligned}$$

$$= \frac{(-1)^n (1+\beta)_n}{(1+\alpha)_n} {}_3F_2 \left[\begin{matrix} -n, 1 + b, 1 + \alpha + \beta + n; \\ 1 + \beta, a + b + 2 \end{matrix} ; 1 \right] \tag{3.2}$$

$$= \frac{(-1)^n (1+a)_n \Gamma(1+\alpha+\beta+2n)}{(1+\alpha)_n \Gamma(1+\alpha+\beta+n)(a+b+2)_n} {}_3F_2 \left[\begin{matrix} -n, -\alpha - n, -a - b - n - 1; \\ -a - n, -\alpha - \beta - 2n \end{matrix} ; 1 \right] \tag{3.3}$$

$$= \frac{(-1)^n (1+\beta)_n (1+a)_n}{(1+\alpha)_n (a+b+2)_n} {}_3F_2 \left[\begin{matrix} -n, -\alpha - n, b + 1 \\ -a - n, \beta + 1 \end{matrix} ; 1 \right] \tag{3.4}$$

$$= \frac{(1+b)_n \Gamma(1+\alpha+\beta+2n)}{(1+\alpha)_n \Gamma(1+\alpha+\beta+n)(a+b+2)_n} {}_3F_2 \left[\begin{matrix} -n, -\beta - n, -a - b - n - 1; \\ -b - n, -\alpha - \beta - 2n \end{matrix} ; 1 \right] \tag{3.5}$$

Proof Comparing the integral (2.1) with the remaining five integrals from (2.2) to (2.6), we can obtain the above five hypergeometric transformations associated with terminating Clausen's series ${}_3F_2$.

4 Special Cases

It is to be noted that the reciprocal of Gamma of non-positive integers is always zero.

Case (i)

Putting $a = \alpha$, $b = \sigma$ in equations (2.1), (2.2), (2.3), (2.4), (2.5), applying Gauss theorem

(1.12) and putting $a = \alpha$, $b = \sigma$ in equation (2.6), applying Saalschütz' summation theorem (1.13), we get

$$\int_{-1}^{+1} (1-x)^{\alpha} (1+x)^{\sigma} P_n^{(\alpha, \beta)}(x) dx = \frac{2^{\alpha+\sigma+1} \Gamma(\sigma+1) \Gamma(\alpha+n+1) \Gamma(\sigma-\beta+1)}{\Gamma(\sigma-\beta-n+1) \Gamma(\alpha+\sigma+n+2) n!} \quad (4.1)$$

$$(Re(\alpha) > -1, \quad Re(\sigma) > -1)$$

which is the correct form of the integral (1.14).

Case (ii)

Putting $a = \rho$, $b = \beta$ in equations (2.1), (2.2), (2.3), (2.4), (2.5), applying Saalschütz' summation theorem (1.13) and putting $a = \rho$, $b = \beta$ in equation (2.6), applying Gauss theorem (1.12), we get

$$\int_{-1}^{+1} (1-x)^{\rho} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) dx = \frac{2^{\rho+\beta+1} \Gamma(\rho+1) \Gamma(\beta+n+1) \Gamma(\alpha-\rho+n)}{n! \Gamma(\alpha-\rho) \Gamma(\rho+\beta+n+2)} \quad (4.2)$$

$$(Re(\rho) > -1, \quad Re(\beta) > -1)$$

which is the known integral [1,p.284(16.4.2)].

Case (iii)

Putting $a = \alpha$ and $b = \beta$ in equations (2.1) and (2.4), applying Gauss theorem (1.12), we get

$$\int_{-1}^{+1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) dx = \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+n+1) \Gamma(\beta+1)}{(n)! \Gamma(\alpha+\beta+n+2) \Gamma(1-n)} \quad (4.3)$$

$$= \begin{cases} 0, & \text{when } n = 1, 2, 3, \dots \\ \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{when } n = 0 \end{cases}$$

where $Re(\alpha) > -1$, $Re(\beta) > -1$.

Case (iv)

Putting $a = \alpha$, $b = \beta$ in equation (2.2) and using combinatorial identity (1.16), we get

$$\int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) dx = \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{(n)! \Gamma(\alpha+\beta+n+2)} \sum_{k=0}^n \frac{(-n)_k}{k!} \quad (4.4)$$

$$= \begin{cases} 0, & \text{when } n = 1, 2, 3, \dots \\ \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{when } n = 0 \end{cases}$$

where $Re(\alpha) > -1$, $Re(\beta) > -1$.

Case (v)

Putting $a = \rho$ and $b = \sigma$ in equation (2.1), we get

$$\int_{-1}^{+1} (1-x)^\rho (1+x)^\sigma P_n^{(\alpha, \beta)}(x) dx$$

$$= \frac{2^{\rho+\sigma+1} \Gamma(\rho+1) \Gamma(\sigma+1)}{\Gamma(\rho+\sigma+2)} \frac{(1+\alpha)_n}{n!} {}_3F_2 \left[\begin{matrix} -n, \alpha + \beta + n + 1, \rho + 1; \\ 1 + \alpha, \quad \rho + \sigma + 2; \end{matrix} \middle| 1 \right] \quad (4.5)$$

$$(Re(\rho) > -1 \text{ , } Re(\sigma) > -1)$$

which is the correct form of the integral (1.15).

Case (vi)

Putting $a = \alpha$, $b = \beta$ in equations (2.3), (2.6) and applying Gauss theorem (1.12) , we get

$$\int_{-1}^{+1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) dx = \frac{(-1)^n 2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+n+1)}{(n)! \Gamma(\alpha+\beta+n+2) \Gamma(1-n)} \quad (4.6)$$

$$= \begin{cases} 0, & \text{when } n = 1, 2, 3, \dots \\ \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{when } n = 0 \end{cases}$$

where $Re(\alpha) > -1$, $Re(\beta) > -1$.

The integral (4.6) can be obtained from the integral (4.3) if we interchange α and β in equation (4.3), use the property of definite integral (1.18) and a result (1.17) on Jacobi Polynomials.

Case (vii)

Putting $a = \alpha$, $b = \beta$ in equation (2.5) and using combinatorial identity (1.16), we get

$$\int_{-1}^{+1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) dx = \frac{(-1)^n 2^{(\alpha+\beta+1)} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{(n)! \Gamma(\alpha+\beta+n+2)} \sum_{k=0}^n \frac{(-n)_k}{k!} \quad (4.7)$$

$$= \begin{cases} 0, & \text{when } n = 1, 2, 3, \dots \\ \frac{2^{(\alpha+\beta+1)} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & \text{when } n = 0 \end{cases}$$

where $Re(\alpha) > -1$, $Re(\beta) > -1$.

The integral (4.7) can also be obtained from the integral (4.4) using the same technique as discussed in case (vi).

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